Principles of Analysis I

Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences. Students in section 500 may substitute problems from part C for problems in part A if they wish.

A Section 500: Do both of these problems.

A.1

Give an example of a metric space that is neither connected, nor totally bounded, nor complete. Say why your example has the required properties.

A.2

Consider the space $C[0,1]$ of continuous functions on the closed interval $[0,1]$ provided with the standard metric: $d(f,g) = \max_{0\le x\le 1} |f(x) - g(x)|$. Let $L\colon C[0,1] \to \mathbb{R}$ be the function defined via $L(f) = \int_0^1 f(x)\sin(x)\,dx$. Prove that L is a continuous function.

Remark This problem is an instance of the general fact that the Fourier coefficients of a function depend continuously on the function.

B Section 500 and Section 200: Do *two* of these problems.

B.1

(a) Is it true in every metric space that every closed set is equal to the closure of its interior? Give either a proof or a counterexample.

(b) Is it true in every metric space that every open set is equal to the interior of its closure? Give either a proof or a counterexample.

B.2

Suppose (M,d) is a complete metric space containing at least two points, and suppose there is a point x_0 in M such that $(M \setminus \{x_0\}, d)$ is a complete metric space too. Prove that M is disconnected.

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B.3

Consider the following two subsets of the real numbers $\mathbb{R}$ equipped with the standard metric: $\mathbb{N}$ is the set of natural numbers 1, 2, 3, …; and S is the set of reciprocals of natural numbers 1, 1/2, 1/3, …. Show that S is totally bounded, $\mathbb{N}$ is *not* totally bounded, and $\mathbb{N}$ is homeomorphic to S.

Remark This example is a special instance of a general property: namely, a metric space is separable if and only if it is homeomorphic to a totally bounded space.

B.4

In the sequence space ℓ_2 [the space of sequences $x = (x_1, x_2, \ldots)$ with norm $\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$], let S denote the set of absolutely summable sequences [that is, sequences $(x_1, x_2, \ldots)$ for which the series $\sum_{n=1}^{\infty} |x_n|$ converges]. Prove that S is dense in ℓ_2 [that is, the closure of S equals the whole space ℓ_2].

C Section 200: Do both of these problems.

C.1

Suppose $f\colon (M, d) \to (N, \rho)$ is a function between metric spaces with the property that for every convergent sequence (x_n) in M, the image sequence $(f(x_n))$ has a convergent *subsequence*. Must f be continuous? Supply a proof or a counterexample, as appropriate.

C.2

Connie conjectures that the following statement holds in every complete metric space: If (F_n) is a decreasing sequence of nonempty nested sets (that is, $F_1 \supset F_2 \supset F_3 \supset \cdots$), if the set F_n is both closed and connected for every n, and if the intersection $\bigcap_{n=1}^{\infty} F_n$ is nonempty, then the intersection $\bigcap_{n=1}^{\infty} F_n$ must be connected. Either prove Connie's conjecture or give a counterexample, as appropriate.