Principles of Analysis I

Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences.

A Section 500: Do both of these problems.

A.1

In this course, you learned various "c" notions. Some of these concepts are (a) countable, (b) closed, (c) connected, (d) compact, and (e) first category. The Cantor set (viewed as a subset of the real numbers with the standard metric) has which of these properties? Why?
[You may substitute problem C.1 for problem A.1 if you wish.]

Solution.

(a) The uncountability of the Cantor set is a proposition that we proved. (The points of the Cantor set have ternary expansions consisting of 0's and 2's; halving the digits produces all the binary expansions of 0's and 1's, so we have a surjection from the Cantor set onto the real numbers between 0 and 1, a set that we know to be uncountable by Cantor's diagonal argument.)

(b) The Cantor set is closed because its complement (a union of open intervals) is open.

(c) The Cantor set is disconnected since it is covered by the two disjoint open intervals $(-1, 1/2)$ and $(1/2, 2)$, each of which has nonempty intersection with the Cantor set. Actually, the Cantor set is totally disconnected (its only nonempty connected subsets are singletons) because, by a homework exercise, the Cantor set contains no intervals.

(d) The Cantor set is compact because it is a closed and bounded subset of $\mathbb{R}$.

(e) Viewed as a subset of $\mathbb{R}$, the Cantor set is nowhere dense (it is a closed set containing no intervals), so it is a set of first category. Viewed as a subset of itself, the Cantor set is of second category by the Baire category theorem (since the Cantor set is a complete metric space).

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A.2

Consider the continuous function $f\colon (0,1) \to \mathbb{R}$ defined by $f(x) = \dfrac{\sin(x)}{x}$. Is this continuous function *uniformly* continuous on the open interval $(0,1)$? Explain why or why not.
[You may substitute problem C.2 for problem A.2 if you wish.]

Solution. The function is uniformly continuous on the open interval $(0,1)$. Making a concrete estimate of $\left|\dfrac{\sin(x)}{x} - \dfrac{\sin(y)}{y}\right|$ appears difficult, so you should try to apply some theorem.

One possibility is to compute the derivative $f'(x)$ and show that it is bounded. A function with a bounded derivative satisfies a Lipschitz condition and hence is uniformly continuous.

A simpler method is to observe that $\lim_{x\to 0} f(x) = 1$, and $\lim_{x\to 1} f(x) = \sin(1)$, so f extends to be a continuous function on the closed interval $[0,1]$. We proved a theorem stating that a continuous function on a closed, bounded interval is automatically uniformly continuous.

B Section 500 and Section 200: Do *two* of these problems.

B.1

Suppose $0 \le x \le 1$, and $f_n(x) = \dfrac{x^n}{1+nx}$ when n is a positive integer. Discuss convergence of the sequence (f_n) on the closed interval $[0,1]$. (Does this sequence of functions converge pointwise? uniformly? How do you know?)

Solution. Since $0 \le f_n(x) \le x^n$, we see that $f_n(x) \to 0$ as $n \to \infty$ if $0 \le x < 1$. On the other hand, $f_n(1) = 1/(1+n)$, so $f_n(1) \to 0$ too. Thus the indicated sequence of functions converges pointwise to the 0 function.

One way to show that the convergence is actually uniform is to use a "divide and conquer" method. Observe that

$$\max_{0\le x\le 1/2} |f_n(x) - 0| \le \max_{0\le x\le 1/2} x^n = \frac{1}{2^n},$$

and

$$\max_{1/2\le x\le 1} |f_n(x) - 0| \le \max_{1/2\le x\le 1} \frac{1}{nx} = \frac{2}{n}.$$

Therefore $\max_{0\le x\le 1} |f_n(x) - 0| \to 0$ as $n \to \infty$, so the convergence is uniform. (Indeed, if ε is an arbitrary positive number, and if $n > 2/\varepsilon$, then $\max_{0\le x\le 1} |f_n(x) - 0| < \varepsilon$.)

Alternatively, you could compute the derivative $f_n'(x)$, observe that the derivative is never negative when $0 \le x \le 1$, and deduce that each $f_n(x)$ is an increasing function of x. Therefore $\max_{0\le x\le 1} |f_n(x)-0| = f_n(1) = 1/(1+n)$. Since this maximum tends to 0 as n increases, the convergence of the sequence of functions is uniform.

Since for each x the sequence $(f_n(x))$ is monotonically decreasing as n increases, the uniformity of the convergence also follows from Dini's theorem (Exercise 18 on page 151 in the textbook), but I did not expect you to solve the problem that way since we did not do that exercise.

B.2

In the metric space $C[0, 1]$ of continuous real-valued functions on the closed interval $[0, 1]$, let S be the subset consisting of those continuous functions f such that $f(0) = 0$ and $|f(x) - f(y)| \le |x - y|$ for all x and y. Is the set S a *compact* subset of $C[0, 1]$? Explain.

Solution. According to the Arzelà–Ascoli theorem, we will know that the set S is a compact subset of $C[0, 1]$ if we can check that the set S is (a) closed, (b) pointwise bounded, and (c) uniformly equicontinuous.

That S is closed follows because the conditions defining S are preserved under taking limits. More precisely, if (f_n) is a sequence in S converging to some limit function f in $C[0, 1]$, then $|f(x)-f(y)| = \lim_{n\to\infty} |f_n(x)-f_n(y)| \le |x - y|$, and $f(0) = \lim_{n\to\infty} f_n(0) = 0$.

That S is pointwise bounded follows because if $f \in S$ and $x \in [0, 1]$, then $|f(x)| = |f(x)-f(0)| \le |x-0| \le 1$. Thus S is even uniformly bounded by 1.

That S is uniformly equicontinuous follows from the hypothesis that f satisfies a Lipschitz condition. Namely, if ε is an arbitrary positive number, we can take δ equal to ε. Then if $|x - y| < \delta$, it follows from the Lipschitz condition that $|f(x) - f(y)| < \varepsilon$ for every f in S. Since the same δ works for every f and for all x and y, the set S of functions is uniformly equicontinuous.

B.3

State the following three theorems: (a) the Bolzano–Weierstrass theorem for real numbers, (b) Hölder's inequality for sequences, and (c) the Weierstrass approximation theorem (any version).

Solution.

(a) The Bolzano–Weierstrass theorem says that every bounded sequence of real numbers has a convergent subsequence; or, equivalently, that every bounded infinite subset of $\mathbb{R}$ has a limit point.

(b) Hölder's inequality says that if p and q are real numbers greater than 1 such that $p^{-1} + q^{-1} = 1$, if $x \in \ell_p$, and if $y \in \ell_q$, then $\sum_{j=1}^{\infty} |x_j y_j| \leq \|x\|_p \|y\|_q$.

(c) The Weierstrass approximation theorem says that the polynomials are dense in $C[0, 1]$; or, in its trigonometric version, that the trigonometric polynomials are dense in $C[0, 1]$.

B.4

Suppose (M, d) and (N, ρ) are homeomorphic metric spaces. If M is complete, must N be complete? If M is separable, must N be separable? Explain.

Solution. The answer to the first question is negative: take M to be the complete metric space $\mathbb{R}$ and take N to be the incomplete metric space $(0, 1)$ (both with the standard metric). These spaces are homeomorphic [via $x \mapsto (1/2) + \pi^{-1} \arctan(x)$].

The answer to the second question is affirmative. Recall that a metric space is separable if it has a countable dense subset. A homeomorphism is, in particular, a bijection, so a homeomorphism takes a countable set to a countable set. Moreover, homeomorphisms preserve convergent sequences, open sets, and closed sets, so homeomorphisms take dense sets to dense sets.

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C Section 200: Do both of these problems.

C.1

In this course, you learned various "c" notions. Some of these concepts are (a) countable, (b) closed, (c) connected, (d) compact, and (e) first category. The set of sequences of 0's and 1's, viewed as a subset of the normed space ℓ_∞ of bounded sequences, has which of these properties? Why?

Solution. Call the indicated set S.

(a) The set S of sequences of 0's and 1's is equivalent to the set of real numbers between 0 and 1, so S is an uncountable set.

(b) The set S is closed because the only convergent sequences of points of S are eventually constant (because the distance between two distinct elements of S equals 1). Hence S contains all its limit points.

(c) The set S is disconnected because there exists a continuous function from S onto the two-point discrete space $\{0, 1\}$. Indeed, every function from S into another metric space is continuous because all the points of S are isolated. One example of a surjective function from S onto $\{0, 1\}$ is the function that takes the sequence $(0, 0, 0, \ldots)$ to 0 and every other sequence to 1. In fact, the set S is totally disconnected.

(d) The set S is not compact because it is not totally bounded. An open ball in ℓ_∞ of radius $1/3$ contains at most one element of S, so no finite number of such balls can cover S.

(e) The set S, viewed as a subset of ℓ_∞, is nowhere dense (it is a closed set that contains no open ball), so it is a set of first category. Viewed as a subset of itself, the set S is a set of second category by the Baire category theorem (since S is a complete metric space).

C.2

Let M be the metric space $\mathbb{R}^2 \setminus \{(0, 0)\}$ (the "punctured plane") equipped with the standard metric inherited from $\mathbb{R}^2$. Consider the continuous func-

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tion $f\colon M \to \mathbb{R}$ defined by $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Is this function *uniformly* continuous on M? Explain why or why not.

Solution. We can see that f is *not* uniformly continuous by exhibiting two sequences (p_n) and (q_n) in M such that $\|p_n - q_n\| \to 0$ but $|f(p_n) - f(q_n)| = 2$ (for then the definition of uniform continuity cannot be satisfied with ε equal to 1). Take $p_n = (1/n, 0)$ and $q_n = (0, 1/n)$. Then $\|p_n - q_n\| = \sqrt{2}/n \to 0$, and $f(p_n) - f(q_n) = 2$.

D Extra credit (optional) for both Section 500 and Section 200

For bonus points, prove either (a) the Bernstein equivalence theorem about sets of the same cardinality or (b) the Baire category theorem for complete metric spaces.

Solution. These proofs can be found in the textbook on pages 24 and 131–133.