Principles of Analysis II

Instructions Solve *four* of the following six problems. Please write your solutions on your own paper. Explain your reasoning in complete sentences.

1. Suppose that A and B are two Borel subsets of the real numbers $\mathbb{R}$. Prove that Lebesgue measure m satisfies the following property:

$$m(A \cap B) + m(A \cup B) = m(A) + m(B).$$

Solution. The hypothesis that A and B are Borel sets is needed only to guarantee that A and B are (Lebesgue) measurable sets. The intersection $A \cap B$ and the union $A \cup B$ are then measurable sets, and so are the set differences $A \setminus B$ and $B \setminus A$.

The measure of the union of two disjoint measurable sets is the sum of the measures, and $A = (A \setminus B) \cup (A \cap B)$, so $m(A) = m(A \setminus B) + m(A \cap B)$. Similar considerations hold for the set B, so

$$m(A) + m(B) = m(A \setminus B) + 2m(A \cap B) + m(B \setminus A). \tag{1}$$

Moreover, $A \cup B$ is the union of the pairwise disjoint sets $A \setminus B$, $B \setminus A$, and $A \cap B$, so $m(A \cup B) = m(A \setminus B) + m(B \setminus A) + m(A \cap B)$. Therefore

$$m(A \cap B) + m(A \cup B) = m(A \setminus B) + 2m(A \cap B) + m(B \setminus A). \tag{2}$$

Since equations (1) and (2) have the same right-hand side, the left-hand sides are equal; thus the required relation holds.

[This problem is essentially Exercise 16.40 on page 281 in the textbook.]

2. Suppose that (E_n) is a sequence of measurable subsets of $\mathbb{R}$. Prove that

$$m\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k\right) \le \liminf_{n\to\infty} m(E_n).$$

Solution. The intersection $\bigcap_{k=n}^{\infty} E_k$ is a subset of E_j when $j \ge n$, so

$$m\left(\bigcap_{k=n}^{\infty} E_k\right) \le m(E_j) \qquad \text{when } j \ge n,$$

and therefore

$$m\left(\bigcap_{k=n}^{\infty} E_k\right) \le \liminf_{j\to\infty} m(E_j).$$

When n increases, the intersection $\bigcap_{k=n}^{\infty} E_k$ can only get bigger. The measure of a union of increasing sets equals the limit of the measures (by the "continuity property of Lebesgue measure" in Theorem 16.23(i) on page 284), so

$$m\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k\right) = \lim_{n\to\infty} m\left(\bigcap_{k=n}^{\infty} E_k\right) \le \liminf_{j\to\infty} m(E_j).$$

If "lim inf" makes you think of Fatou's lemma, you are right, and you could solve the problem in a different way as follows. To simplify the writing, let S denote the set $\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k$. A point x is in the set S if and only if x is in all but finitely many of the sets E_k. A reinterpretation of this statement in terms of characteristic functions is that $\chi_S(x) = \liminf_{k\to\infty} \chi_{E_k}(x)$. Fatou's lemma then implies that

$$m(S) = \int \chi_S = \int \liminf_{k\to\infty} \chi_{E_k} \le \liminf_{k\to\infty} \int \chi_{E_k} = \liminf_{k\to\infty} m(E_k).$$

[This problem, part of Exercise 16.62 on page 286 in the textbook, is a counterpart of the Borel–Cantelli lemma, which is stated in Corollary 16.24.]

3. Prove that every increasing function $f\colon \mathbb{R} \to \mathbb{R}$ is measurable.

   [Notice that f need not be continuous.]

   Solution. The statement is part of Corollary 17.3 on page 297 of the textbook, but the author gives no proof, saying only that the property is "easily seen."

   What needs to be shown is that for every real number α, the inverse image set $\{\, x \in \mathbb{R} : f(x) > \alpha \,\}$ is a measurable set. If this set happens to be either the empty set $\varnothing$ or the whole set $\mathbb{R}$, then there is nothing to show. If neither of these special cases occurs, then this inverse image set has some finite number β as infimum. Because the function f is increasing, this inverse image set is either (β,∞) or $[\beta,\infty)$. Hence the set $\{\, x \in \mathbb{R} : f(x) > \alpha \,\}$ is indeed measurable (being either an open set or a closed set).

4. Give an example of a sequence (f_n) of measurable functions (from $\mathbb{R}$ into $\mathbb{R}$) converging pointwise to a limit function f but not converging almost uniformly.

   [Such an example shows that the conclusion of Egorov's theorem fails on unbounded intervals. Recall that $f_n \to f$ almost uniformly if for every positive ε there exists a measurable set E of measure less than ε such that $f_n \to f$ uniformly on the complement of E.]

   Solution. Let f_n be the characteristic function of the interval $[n, n+1]$. Then the sequence (f_n) converges pointwise to the zero function when $n \to \infty$. The convergence fails to be almost uniform, however, because $|f_n(x) - f(x)| = 1$ for every x in the interval $[n, n+1]$, a set of measure 1. Thus the definition of almost uniform convergence is violated when $\varepsilon = 1/2$.

5. Give an example of a sequence (φ_n) of nonnegative simple functions such that $\int \varphi_n \le 1$ for every n, but $\int \varphi_n^2 \to \infty$ as $n \to \infty$.

   Solution. Let φ_n be n times the characteristic function of the interval $[0, 1/n]$. Then $\int \varphi_n = \int_0^{1/n} n = 1$ for every n, but $\int \varphi_n^2 = \int_0^{1/n} n^2 = n \to \infty$.

6. Apply an appropriate convergence theorem for integrals to compute

$$\lim_{n\to\infty} \int_0^1 \frac{n\cos(x)}{1+n^2x^2}\,dx.$$

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Solution. Substituting t for nx shows that

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 \frac{n\cos(x)}{1+n^2x^2}\,dx &= \lim_{n\to\infty}\int_0^n \frac{\cos(t/n)}{1+t^2}\,dt\\ &= \lim_{n\to\infty}\int_{\mathbb{R}} \frac{\chi_{[0,n]}(t)\cos(t/n)}{1+t^2}\,dt.\end{aligned}$$

Since the absolute value of the integrand is bounded above for every n by the integrable function $1/(1+t^2)$, Lebesgue's dominated convergence theorem applies. Taking the limit inside the integral sign and observing that $\lim_{n\to\infty}\cos(t/n) = \cos(0) = 1$ shows that

$$\lim_{n\to\infty}\int_0^1 \frac{n\cos(x)}{1+n^2x^2}\,dx = \int_0^\infty \frac{1}{1+t^2}\,dt = \frac{\pi}{2}.$$