Principles of Analysis II

Instructions Solve *four* of the following six problems. Please write your solutions on your own paper. Explain your reasoning in complete sentences.

1. Let f be a function of bounded variation on the interval $[0, 1]$. Suppose there is a positive number δ such that $|f(x)| \geq \delta$ for every x (in other words, the function f is bounded away from 0). Show that the reciprocal function $1/f$ is a function of bounded variation.

   Solution. Suppose $0 = x_0 < x_1 < \cdots < x_n = 1$. These points x_k define a partition P of the interval $[0, 1]$. The quantity $V(1/f, P)$, the variation of the function $1/f$ with respect to the partition P, equals

   $$\sum_{k=1}^{n} \left| \frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})} \right|.$$

   Observe that

   $$\left| \frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})} \right| = \frac{|f(x_k) - f(x_{k-1})|}{|f(x_k)f(x_{k-1})|} \leq \frac{1}{\delta^2}|f(x_k) - f(x_{k-1})|$$

   for each k. Therefore $V(1/f, P) \leq \delta^{-2}V(f, P) \leq \delta^{-2}V_0^1(f)$, where $V_0^1(f)$ denotes the total variation of f. Taking the supremum over all partitions P shows that the function $1/f$ has bounded variation, and moreover the total variation of this function does not exceed $\delta^{-2}V_0^1(f)$.

2. Give a concrete example of a uniformly convergent sequence (f_n) of functions of bounded variation on the interval $[0, 1]$ such that the limit function does not have bounded variation.

   Solution. You saw in class the standard example of a continuous function f that does not have bounded variation: namely,

   $$f(x) = \begin{cases} x\sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

   To make use of this example, define f_n as follows:

   $$f_n(x) = \begin{cases} x\sin(1/x), & \text{if } x > 1/n, \\ 0, & \text{if } x \leq 1/n. \end{cases}$$

The function f_n has bounded variation because the graph consists of finitely many bounded, monotonic pieces. (The number of pieces grows with n, however.) The sequence (f_n) converges uniformly to f because

$$\sup_{0 \le x \le 1} |f_n(x) - f(x)| \le 1/n.$$

3. If α is a nondecreasing function on the closed interval $[-\pi, \pi]$, is it necessarily true that $\lim_{n\to\infty} \int_{-\pi}^{\pi} \cos(nx)\, d\alpha(x) = 0$? (In other words, does the Riemann–Lebesgue lemma carry over to the setting of the Stieltjes integral?) Give either a proof or a counterexample.

   Solution. For a counterexample, consider the step function α such that $\alpha(x) = 0$ if $x < 0$, and $\alpha(x) = 1$ if $x \ge 0$. You know from class and from the homework exercises that Stieltjes integration against this α acts like a "delta function": in other words, $\int_{-\pi}^{\pi} \cos(nx)\, d\alpha(x) = \cos(0) = 1$ for every n. Hence $\lim_{n\to\infty} \int_{-\pi}^{\pi} \cos(nx)\, d\alpha(x) \ne 0$.

   Remark If α is differentiable and has a derivative that is Riemann integrable, then $\int_{-\pi}^{\pi} \cos(nx)\, d\alpha(x) = \int_{-\pi}^{\pi} \cos(nx)\alpha'(x)\, dx$ [by Theorem 14.18 on page 232]; in this case, the limit of the integral *is* equal to 0 by the usual Riemann–Lebesgue lemma [equation (15.4) on page 248 or Theorem 19.17 on page 353].

4. Let f be a bounded function that is Riemann–Stieltjes integrable with respect to the increasing function α on the interval $[0, 1]$. Prove that f is Riemann–Stieltjes integrable with respect to α^2 on the same interval. In other words, if $\int_0^1 f\, d\alpha$ exists, then so does $\int_0^1 f\, d(\alpha^2)$.

   Remark added May 12 The solution below assumes that the function α^2 is increasing, which need not be the case. [The function $\alpha(x)$ might be $x - 1/2$, for example.] There are a couple of ways to reduce to the situation in which α^2 is increasing.

   1. Either α is nonnegative (in which case α^2 is increasing), or α is nonpositive (in which case α^2 is decreasing, and $-\alpha^2$ is increasing), or there is a point c where α changes sign from negative to positive (in which case α^2 is decreasing on the interval $[0, c]$ and increasing on the

interval $[c, 1]$). In the first case, the argument below applies. In the second case, the argument below shows that $\int_0^1 f\,d(-\alpha^2)$ exists, and it follows from the definitions that $\int_0^1 f\,d(-\alpha^2) = -\int_0^1 f\,d(\alpha^2)$. In the third case, use the preceding two cases to deduce that both $\int_0^c f\,d(\alpha^2)$ and $\int_c^1 f\,d(\alpha^2)$ exist; hence $\int_0^1 f\,d(\alpha^2)$ exists.

2. Alternatively, observe that there is a constant k for which the function $\alpha + k$ is positive (any value of k greater than $|\alpha(0)|$ will do). The argument below implies that $\int_0^1 f\,d((\alpha+k)^2)$ exists. It is routine to see that existence of $\int_0^1 f\,d\alpha$ implies existence of $\int_0^1 f\,d(2k\alpha + k^2)$ [which equals $2k\int_0^1 f\,d\alpha$], and the difference $\int_0^1 f\,d((\alpha+k)^2) - \int_0^1 f\,d(2k\alpha+k^2)$ equals $\int_0^1 f\,d(\alpha^2)$.

Solution. Fix an arbitrary positive ε. The integrability of f with respect to α implies (by Riemann's condition) that there is a partition P such that the upper sum $U_\alpha(f, P)$ and the lower sum $L_\alpha(f, P)$ differ by less than ε. In other words, there is a subdivision of the interval $[0, 1]$ into n subintervals $[x_{k-1}, x_k]$ such that if M_k and m_k denote the supremum and the infimum of f on $[x_{k-1}, x_k]$, then

$$\sum_{k=1}^{n}(M_k - m_k)[\alpha(x_k) - \alpha(x_{k-1})] = U_\alpha(f, P) - L_\alpha(f, P) < \varepsilon.$$

The goal is to estimate the difference between upper and lower sums with respect to α^2. The function α is increasing, so $\alpha(x_k)^2 - \alpha(x_{k-1})^2 = [\alpha(x_k) - \alpha(x_{k-1})][\alpha(x_k) + \alpha(x_{k-1})] \le 2\alpha(1)[\alpha(x_k) - \alpha(x_{k-1})]$. Therefore

$$\begin{aligned} U_{\alpha^2}(f, P) - L_{\alpha^2}(f, P) &= \sum_{k=1}^{n}(M_k - m_k)[\alpha(x_k)^2 - \alpha(x_{k-1})^2] \\ &\le 2\alpha(1)[U_\alpha(f, P) - L_\alpha(f, P)] \le 2\varepsilon\alpha(1). \end{aligned}$$

Since ε is arbitrary, Riemann's condition implies that f is integrable with respect to α^2.

5. Determine the Fourier series of the odd function on the interval $[-\pi, \pi]$ that is equal to 1 on the interval $(0, \pi)$, and use the result to compute the value of the numerical series $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$.

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Solution. The Fourier series of an odd function is a sine series of the form $\sum_{n=1}^{\infty} b_n \sin(nx)$. For the specified function,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi 1\cdot \sin(nx)\,dx = \frac{2}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^\pi \\ &= \frac{2}{n\pi}[-\cos(n\pi)+\cos(0)] = \begin{cases} \frac{4}{n\pi}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Writing an odd integer n in the form $2k+1$ shows that the Fourier series has the form

$$\sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi}\sin[(2k+1)x].$$

According to Parseval's equation, the sum of the squares of the Fourier coefficients equals $1/\pi$ times the integral of the square of the function over the interval $[-\pi,\pi]$. Thus

$$\frac{16}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{2}{\pi}\int_0^\pi 1^2\,dx = 2, \quad \text{so} \quad \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Remark You summed this series using a different Fourier series in Assignment 8.

6. Suppose $f \in L_2[-\pi,\pi]$. Then $s_n(f)$, the nth partial sum of the Fourier series of f, has the property that $\lim_{n\to\infty} \|s_n(f)-f\|_2 = 0$ (according to the Riesz–Fischer theorem). Use this result to prove that the Cesàro sum $\sigma_n(f)$, which is the average $[s_0(f)+\cdots+s_{n-1}(f)]/n$, has the corresponding property that $\lim_{n\to\infty}\|\sigma_n(f)-f\|_2 = 0$.

Solution. Fix an arbitrary positive ε. By the Riesz–Fischer theorem, there is a positive integer N such that $\|s_k(f)-f\|_2 < \varepsilon/2$ when $k \ge N$.

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Thus when $n > N$, the definition of $\sigma_n(f)$ implies that

$$\begin{aligned}\|\sigma_n(f) - f\|_2 &= \left\| \frac{[s_0(f) - f] + \cdots + [s_{n-1}(f) - f]}{n} \right\|_2 \\ &\le \frac{1}{n} \sum_{k=0}^{N-1} \|s_k(f) - f\|_2 + \frac{1}{n} \sum_{k=N}^{n-1} \|s_k(f) - f\|_2 \\ &< \frac{\varepsilon}{2} + \frac{1}{n} \sum_{k=0}^{N-1} \|s_k(f) - f\|_2.\end{aligned}$$

Since N is fixed (dependent on ε but not on n), the second term in the third line of the displayed formula will be less than $\varepsilon/2$ when n is sufficiently large. Hence $\|\sigma_n(f) - f\|_2 < \varepsilon$ when n is sufficiently large. Therefore $\lim_{n\to\infty} \|\sigma_n(f) - f\|_2 = 0$.

Bonus problem For extra credit, prove either the Riesz representation theorem characterizing the dual space of $C[0, 1]$ or Jordan's decomposition theorem for functions of bounded variation.

Solution. We did these proofs in class, and there are proofs in the book too (pages 237–239 and 207).