1. Let E denote the open subset of the complex plane defined by $E := \{\, z \in \mathbb{C} : |\sin(z)| < |z| \,\}$. Show that the area of the set E is infinite.

Solution. By the triangle inequality, $|\sin z| \le \frac{1}{2}(|e^{iz}| + |e^{-iz}|) = \frac{1}{2}(e^{-y} + e^{y}) \le e^{|y|}$. Therefore, if $0 < y < 1$ and $x > 3$, we have $|\sin z| < e < x < |z|$. Hence the set E contains an unbounded half-strip of height 1, so E certainly has infinite area.

2. Solve exercise 2.4 in the textbook: namely, derive the Cauchy-Riemann equations in polar coordinates.

Solution. If the derivative f' exists as a two-dimensional limit, then on the one hand $f'(z)$ equals

$$\lim_{h\to 0} \frac{f((r+h)e^{i\theta}) - f(re^{i\theta})}{he^{i\theta}} = e^{-i\theta}\frac{\partial f}{\partial r}(z),$$

and on the other hand $f'(z)$ equals

$$\lim_{\psi\to 0} \frac{f(re^{i(\theta+\psi)}) - f(re^{i\theta})}{re^{i\theta}(e^{i\psi} - 1)} = \frac{e^{-i\theta}}{r}\frac{\partial f}{\partial \theta}(z)\frac{1}{\partial e^{i\psi}/\partial\psi(0)} = \frac{-ie^{-i\theta}}{r}\frac{\partial f}{\partial \theta}(z).$$

Equating the two expressions shows that $\frac{\partial f}{\partial r} = \frac{-i}{r}\frac{\partial f}{\partial \theta}$. Taking real and imaginary parts reveals that $\frac{\partial U}{\partial r} = \frac{1}{r}\frac{\partial V}{\partial \theta}$ and $\frac{\partial V}{\partial r} = -\frac{1}{r}\frac{\partial U}{\partial \theta}$.

Alternatively, one can start from the Cauchy-Riemann equations in rectangular coordinates and apply the chain rule:

$$\begin{aligned}
\frac{\partial U}{\partial r} &= \frac{\partial U}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial U}{\partial x}\cos\theta + \frac{\partial U}{\partial y}\sin\theta, \\
\frac{\partial V}{\partial \theta} &= \frac{\partial V}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial V}{\partial x}r\sin\theta + \frac{\partial V}{\partial y}r\cos\theta \\
&= \frac{\partial U}{\partial y}r\sin\theta + \frac{\partial U}{\partial x}r\cos\theta.
\end{aligned}$$

Hence $\frac{\partial V}{\partial \theta} = r\frac{\partial U}{\partial r}$; and similarly for the second of the Cauchy-Riemann equations.

3. We know that a power series converges absolutely in a certain disk. Consider instead a *Dirichlet series* of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z}, \tag{1}$$

where the complex numbers a_n are constants (independent of the variable z), and the expression n^z means, by definition, $\exp(z \ln n)$, where ln denotes the natural logarithm of a positive real number. Set

$$A := \limsup_{n\to\infty} \frac{\ln |a_n|}{\ln n}.$$

Supposing that A is finite, show that the Dirichlet series (1) converges absolutely when $\operatorname{Re} z > A + 1$.

Solution. With no extra work, one can show that the convergence is absolute and uniform in any half-plane where $\operatorname{Re} z \geq B > A + 1$. Indeed, let C be a number such that $B > C > A + 1$. Then there is a number N such that $\frac{\ln |a_n|}{\ln n} < C - 1$ when $n > N$. Now $|a_n| < n^{C-1}$ for such n, which means that

$$\left|\frac{a_n}{n^z}\right| < \frac{n^{C-1}}{n^B} = \frac{1}{n^{B-C+1}}.$$

Since $B - C + 1 > 1$, the series $\sum_n 1/n^{B-C+1}$ converges, and so the dominated series $\sum_n a_n/n^z$ converges absolutely and uniformly in the indicated closed half-plane $\{\, z : \operatorname{Re} z \geq B \,\}$.

4. The power series $1 - z + z^2 - z^3 + \cdots$ is a geometric series that converges to $1/(1 + z)$ when $|z| < 1$. Consequently, one expects that the formal anti-derivative

$$L(z) := z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

should have the properties of a logarithm of $1 + z$. Prove that indeed $\exp(L(z)) = 1 + z$ when $|z| < 1$.

You may assume that a power series can be differentiated term by term inside the open disk of convergence (a fact that we have stated but not yet officially proved).

Solution. Let $f(z)$ denote the function $(1+z)e^{-L(z)}$. Then $f'(z) = e^{-L(z)} - (1+z)e^{-L(z)}L'(z) = 0$. Hence f is a constant function, so $ce^{L(z)} = 1+z$. When $z = 0$ one finds that $c = 1$. Hence $e^{L(z)} = 1+z$.

5. Let C be a continuously differentiable simple closed curve equipped with the standard counterclockwise orientation. Show that $\int_C (\operatorname{Im} z)\, dz$ equals the negative of the area of the region enclosed by the curve C.

Solution. Green's theorem in complex form says that

$$\int_C f(z)\, dz = 2i \iint \frac{\partial f}{\partial \overline{z}}\, dx\, dy.$$

Since $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z}\,)$, one has that $\frac{\partial \operatorname{Im} z}{\partial \overline{z}} = \frac{-1}{2i}$. Inserting this information into Green's theorem shows that

$$\int_C \operatorname{Im} z\, dz = \iint -dx\, dy,$$

which indeed equals the negative of the area of the region enclosed by the curve C.

6. Suppose f is an analytic function in the unit disk $\{\, z \in \mathbb{C} : |z| < 1\,\}$. Then, by definition, the function f has a derivative. This problem asks you to show that the function f also *is* a derivative. Namely, set

$$F(z) := \int_0^1 z f(tz)\, dt \qquad \text{when } |z| < 1.$$

Prove that F is differentiable and that $F'(z) = f(z)$.

Solution. Observe that $F(z)$ equals $\int_C f(w)\, dw$, where C is the path parametrized by tz, $0 \le t \le 1$. Cauchy's theorem implies, however, that the integral is independent of the path joining 0 to z. Consequently, $F(z+h) - F(z)$ equals the integral of f along any path joining z to $z+h$, for instance the path parametrized by $z + th$, $0 \le t \le 1$. Therefore

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_0^1 f(z+th)h\, dt = \int_0^1 f(z+th)\, dt.$$

Since f is continuous at z, the integral converges when $h \to 0$ to $\int_0^1 f(z)\, dt$, that is, to $f(z)$. Thus $F'(z)$ exists and equals $f(z)$.