Instructions Do any *five* of the following seven problems.

1. Can a rational function have residue at 0 equal to 1 and residue at infinity equal to 2? Either give an example or prove that none exists.

Solution. One example is $\dfrac{1}{z} - \dfrac{3}{z+1}$.

2. Discuss the equation

$$\overline{\log(z)} = \log(\,\overline{z}\,),$$

where, as usual, the overline indicates complex conjugation. Is the equation always true? In other words, is the complex conjugate of the logarithm always equal to the logarithm of the complex conjugate? If not, what additional conditions will make this true?

Solution. In order for both sides to make sense, one must assume that $z \neq 0$. Then the left-hand side equals $\ln|z| - i \arg z$, and the right-hand side equals $\ln|z| + i \arg \overline{z}$. Hence the two sides are equal if and only if $\arg \overline{z} = -\arg z$. These two quantities are always equal as *sets* of values. If, however, one is dealing with a function (that is, with a specific branch of the logarithm), then equality holds only for the principal branch of the logarithm.

3. The following calculation leads to the absurd conclusion that $L = iL$ (where $L \neq 0$).

$$L := \int_0^\infty \exp(-x^4)\, dx \overset{(x=iu)}{=} \int_0^\infty \exp(-u^4)\, i\, du = iL.$$

Explain the error.

[This is supplementary exercise 2 on page 116.]

Solution. The integrals are improper, so they must be understood as limits. Consider integrating e^{-z^4} over the closed contour in the first quadrant consisting of the segment of the real axis from 0 to R, a quarter circle of radius R, and the segment of the imaginary axis from iR to 0. Since the integrand has no singularities, the integral equals 0 by Cauchy's theorem. Parametrizing the three parts of the contour shows that

$$\int_0^R e^{-x^4}\,dx + \int_0^{\pi/2} \exp(-R^4 e^{4i\theta})\, iRe^{i\theta}\,d\theta = \int_0^R e^{-u^4}\,i\,du.$$

The (false) claim that $L = iL$ is equivalent to the (false) claim that the second integral above tends to 0 as R tends to infinity. There is no reason to expect the second integral to tend to 0, because when $\pi/8 < \theta < 3\pi/8$, the real part of $-e^{4i\theta}$ is positive, so the integrand has exponential growth.

4. A student reasons as follows: "The plane with the half line $(-\infty, 0]$ removed (that is, the set $\mathbb{C} \setminus \{\, z = x + iy : y = 0 \text{ and } x \le 0 \,\}$) is a simply connected region on which the function z^2 has no zeroes. Therefore we can define a branch of $\log(z^2)$ on this region. Since the function z^2 is an *even* function, this branch of $\log(z^2)$ actually makes sense on the domain $\mathbb{C} \setminus \{0\}$: namely, $\log((-z)^2) = \log(z^2)$. Then we can define a branch of $\log(z)$ on $\mathbb{C} \setminus \{0\}$ by setting $\log(z) = \frac{1}{2}\log(z^2)$."

We know that there does *not* exist a global logarithm $\log(z)$ on $\mathbb{C} \setminus \{0\}$, so there is a flaw in the reasoning somewhere. Pinpoint the error.

Solution. Although the notation $\log(f(z))$ looks just like the notation $\sin(f(z))$, the meaning is different. The function $\log f$ is not necessarily the composition of a function "log" with the function f, because log may not be defined globally on the whole range of f. What $\log f$ actually means is either a function whose derivative is f'/f or a function whose exponential is f.

Taking the second point of view, one can see that if $\log(z)$ denotes the principal branch of the logarithm on the slit plane, then the exponential of $2\log(z)$, namely $\exp(2\ln|z| + 2i\arg z)$, equals z^2. Taking the first point of view, one should define $\log(z^2)$ as $\int_1^z (2w/w^2)\,dw$, or $2\log(z)$. From either point of view, it is correct to say that $\log(z^2) = 2\log(z)$ in the slit plane, using the principal branch of $\log(z)$. Evidently the function $\log(z^2)$ is *not* an even function, even though z^2 is an even function.

5. A student reasons as follows: "By symmetry, $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x)}{1+x^2}\,dx = 0$. On the other hand, integrating $\dfrac{\sin(z)}{1+z^2}$ over a closed contour in the upper half plane and passing to the limit shows that the original integral over the real axis equals $2\pi i$ times the residue at $z = i$, or $2\pi i \times \dfrac{\sin(i)}{2i} = \pi \sin(i)$. Therefore $\sin(i) = 0$."

Explain what went wrong.

Solution. The integral over the added piece of contour in the upper half-plane does not tend to 0 in the limit, because the function $\sin(z)$ is unbounded in the upper half-plane. In fact, for every fixed x, we have $\lim_{y\to\infty} |\sin(x+iy)| = \infty$.

The right way to handle the given integral is to treat it as the imaginary part of $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx$.

6. A formula due to Cauchy says that

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)(1-iax)^s}\,dx = \frac{\pi}{(1+a)^s} \qquad (a > 0,\ s > 0).$$

Prove the formula.
[If the positive number s is not an integer, then the power in the denominator of the integral is ambiguous. Assume in this case that the principal branch of the power is taken.]

Solution. Let C_R be the contour consisting of the real axis from $-R$ to R and a semi-circle of (large) radius R in the upper half-plane, and consider

$$\int_{C_R} \frac{1}{(1+z^2)(1-iaz)^s}\,dz.$$

Since z has positive imaginary part inside the contour, and $a > 0$, the quantity $(1 - iaz)$ lies in the right-hand half-plane. Therefore the power $(1-iaz)^s$ can be defined as $\exp\{s \log(1-aiz)\}$ using the principal branch of the logarithm.

Since $a > 0$, the only singularity of the integrand inside the contour is at i. The residue at i equals $\dfrac{1}{2i(1+a)^s}$. By the residue theorem, the integral over C_R equals $\dfrac{2\pi i}{2i(1+a)^s} = \dfrac{\pi}{(1+a)^s}$.

It remains to show that the integral over the semi-circle in the upper half-plane tends to 0 as $R \to \infty$. Since the length of the path equals πR, and (for large R) the integrand is no bigger than $\dfrac{1}{(R^2-1)(aR-1)^s} = O(1/R^{2+s})$, the integral does indeed tend to 0.

7. Prove that

$$\int_0^\infty \frac{1}{1+x^n}\,dx = \frac{\pi/n}{\sin(\pi/n)}, \qquad n = 2, 3, 4, \ldots.$$

Solution. There are two natural ways to compute the integral, one using a pie-shaped contour and one using a keyhole contour.

For the first method, let C_R denote the contour consisting of the real axis from 0 to R, an arc of the circle of radius R from angle 0 to angle $2\pi/n$, and a ray back to the origin at angle $2\pi/n$. Integrate the function $1/(1+z^n)$ over the contour. The only pole inside the contour is at $e^{i\pi/n}$, so the integral over C_R equals

$$\left.\frac{2\pi i}{nz^{n-1}}\right|_{z=e^{i\pi/n}} = \frac{2\pi i}{-ne^{-i\pi/n}}.$$

The integral over the arc is no bigger than $\dfrac{2\pi R/n}{R^n - 1} = O(1/R^{n-1})$. For the ray at angle $2\pi/n$, use the parametrization $z = te^{2\pi i/n}$ to get

$$-e^{2\pi i/n}\int_0^R \frac{1}{1+t^n}\,dt.$$

Putting the pieces together gives

$$\frac{-2\pi i e^{i\pi/n}}{n} = \left(1 - e^{2\pi i/n}\right)\int_0^R \frac{1}{1+x^n}\,dx + O(1/R^{n-1}).$$

Taking the limit as $R \to \infty$ shows that

$$\int_0^\infty \frac{1}{1+x^n}\,dx = \frac{\pi}{n} \cdot \frac{2ie^{i\pi/n}}{e^{2\pi i/n}-1} = \frac{\pi}{n} \cdot \frac{2i}{e^{i\pi/n}-e^{-i\pi/n}} = \frac{\pi/n}{\sin(\pi/n)}.$$

The second method starts by transforming the original real integral by substituting t for x^n:

$$\int_0^\infty \frac{1}{1+x^n}\,dx = \frac{1}{n}\int_0^\infty \frac{t^{-(n-1)/n}}{1+t}\,dt.$$

Because of the fractional power, a keyhole contour is appropriate, using the integrand $\dfrac{z^{-(n-1)/n}}{1+z}$. The contribution from the large circle at radius R is at most $2\pi R$ times $\dfrac{R^{-(n-1)/n}}{R-1}$, and hence tends to 0 as $R \to \infty$. The contribution from the small circle at radius ϵ is at most $2\pi\epsilon$ times $\dfrac{\epsilon^{-(n-1)/n}}{1-\epsilon}$, so this too tends to 0 as $\epsilon \to 0$.

The integral over the real axis "from below" is out of phase relative to the integral over the real axis "from above" by a factor $e^{-(n-1)2\pi i/n} = e^{2\pi i/n}$, so the residue theorem gives

$$\left(1-e^{2\pi i/n}\right)\int_0^\infty \frac{t^{-(n-1)/n}}{1+t}\,dt = 2\pi i(-1)^{-(n-1)/n} = 2\pi i e^{-(n-1)\pi i/n}.$$

Thus

$$\int_0^\infty \frac{1}{1+x^n}\,dx = \frac{2\pi i}{n} \cdot \frac{e^{-\pi i}e^{\pi i/n}}{1-e^{2\pi i/n}} = \frac{2\pi i}{n} \cdot \frac{1}{e^{\pi i/n}-e^{-\pi i/n}},$$

which simplifies as before to the desired answer.