Variations on the maximum principle

I. Hadamard's three-circles theorem

Suppose f is holomorphic in an open annulus $\{ z \in \mathbb{C} : r_1 < |z| < r_2 \}$ and continuous in the closed annulus. Let $M(r)$ denote $\sup\{ |f(z)| : |z| = r \}$. Then $M(r) \le \max(M(r_1), M(r_2))$ when $r_1 \le r \le r_2$ (by the maximum modulus theorem).

Hadamard's three-circles theorem says that more is true: namely, $M(r)$ satisfies a convexity property. The property may be written in either of the following equivalent forms.

$$M(r) \le M(r_1)^{\alpha} M(r_2)^{1-\alpha}, \qquad \text{where } \alpha = \frac{\log(r_2/r)}{\log(r_2/r_1)}. \tag{1}$$

$$\log M(r) \le \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2). \tag{2}$$

In words, the inequality says that $\log M(r)$ is a convex function of $\log r$.

The three-circles theorem can be proved in more than one way, but each method requires overcoming a minor technical difficulty.

1. Prove the three-circles theorem by examining the function $z^{\beta} f(z)$, where the real number β is chosen such that $r_1^{\beta} M(r_1) = r_2^{\beta} M(r_2)$.

Here the technical difficulty is that when β is not an integer, the function $z^{\beta} f(z)$ is locally defined but not globally defined. Nonetheless, one can deduce from the maximum modulus theorem that the globally defined function $|z|^{\beta}|f(z)|$ takes its maximum on the boundary.

2. Prove the three-circles theorem by observing that the right-hand side of inequality (2) defines a harmonic function that dominates the subharmonic function $\log|f(z)|$ on the boundary of the annulus.

Here the technical difficulty is that we declared subharmonic functions to be continuous, but $\log|f(z)|$ is not continuous if f has zeroes. (Some authors allow subharmonic functions to be only upper semi-continuous, in which case the functions may take the value $-\infty$ at some points.) One way to overcome the difficulty is to take the maximum of $\log|f(z)|$ with a suitable negative constant.

3. Equality occurs in the inequalities (1) and (2) for which non-constant holomorphic functions?

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II. Phragmén-Lindelöf theory

4. The exponential function $\exp(z)$ has modulus equal to 1 on the boundary of the right half-plane $\{ z \in \mathbb{C} : \operatorname{Re} z > 0 \}$, but the exponential function is not bounded in the right half-plane. Why does this not contradict the maximum modulus theorem?

The following theorem may be interpreted as saying that the exponential function is the "smallest" counterexample function in the right half-plane. The theorem is the simplest instance of a general technique (based on damping functions) introduced in 1908 by E. Lindelöf and E. Phragmén.

Theorem 1. *Suppose f is holomorphic in the open right half-plane, continuous in the closed right half-plane, and $|f(z)| \le 1$ when $\operatorname{Re} z = 0$. If there exist a real number α strictly less than 1 and constants A and B such that $|f(z)| \le A \exp(B|z|^\alpha)$ when $\operatorname{Re} z > 0$, then $|f(z)| \le 1$ when $\operatorname{Re} z > 0$.*

5. Prove Theorem 1 by examining the function $f(z)\exp(-\epsilon z^\beta)$, where $\alpha < \beta < 1$. Apply the maximum principle on large semi-circles, and let $\epsilon \to 0^+$.

Another instance of the Phragmén-Lindelöf method is a version of the maximum principle with an exceptional boundary point.

Theorem 2. *Let f be a holomorphic function on a bounded domain G in $\mathbb{C}$, and let p be a point of the boundary ∂G. Suppose that $\limsup_{z\to p} |f(z)| < \infty$, and $\limsup_{z\to w} |f(z)| \le 1$ for every point w in $\partial G \setminus \{p\}$. Then $|f(z)| \le 1$ for all z in G.*

6. Prove Theorem 2 by applying the maximum principle to the subharmonic function $|f(z)| + \epsilon \log|z - p|$ and letting $\epsilon \to 0^+$.

7. Take the domain G to be the unit disc, and take the exceptional point p to be 1. Why is the function $\exp((1+z)/(1-z))$ not a counterexample to Theorem 2?