

Instructions Do any five of the following six problems. You may consult the textbook but not other sources; in particular, you may not ask another person for help solving the problems. You may cite and use results proved in the textbook or in class. Please submit your solutions to me in my office (Milner 202) before 4:00 P.M. on Friday, February 16.

1. Solve problem 7 on the May 2004 complex analysis qualifying examination. The statement of the problem reads as follows:

Let f be an entire function. Prove that if f takes purely real values on some pair of lines intersecting at an angle of $\pi/\sqrt{2}$ radians, then f is a constant function.

2. This is a problem about Laurent series expansions “at infinity.”

Let f be an analytic function in a region in \mathbb{C} that contains the complement of some disk. Thus the region is “a punctured neighborhood of infinity.” Then f has a Laurent series in powers of z and z^{-1} that converges when $|z|$ is sufficiently large (that is, when z is “near infinity”). Suppose additionally that f is a one-to-one function onto another punctured neighborhood of infinity, and let g denote the inverse function.

Suppose that the Laurent series of f near infinity has the following form:

$$f(z) = a_1 z + a_0 + \sum_{n=1}^{\infty} \frac{a_{-n}}{z^n}, \quad \text{where } a_1 \neq 0.$$

Show first of all that the Laurent series near infinity of the inverse function g has the following form:

$$g(z) = b_1 z + b_0 + \sum_{n=1}^{\infty} \frac{b_{-n}}{z^n}.$$

In other words, show that all the coefficients of positive powers of z , except for the coefficient b_1 , are equal to 0.

Next show that the first few coefficients b_1 , b_0 , and b_{-1} satisfy the following equations:

$$b_1 = \frac{1}{a_1}, \quad b_0 = -\frac{a_0}{a_1}, \quad \text{and} \quad b_{-1} = -a_{-1}.$$

3. This question asks whether an analog of Pringsheim's lemma holds for Laurent series.

Suppose that $0 < r < R$, and

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad \text{when } r < |z| < R.$$

Suppose that the indicated Laurent series converges in the annulus where $r < |z| < R$ but in no larger concentric open annulus. If every coefficient c_n is a non-negative real number, must the points $z = r$ and $z = R$ both be singular points of f ? [Here "singular" means that f does not admit a direct analytic continuation to a neighborhood of the point.]

Supply either a proof or a counterexample.

4. This problem concerns branches of $\sqrt{z^2 - 1}$, in other words, functions $f(z)$ such that $f(z)^2 = z^2 - 1$.

There are two parts to the question. Each part describes a function $f(z)$ that is a branch of $\sqrt{z^2 - 1}$ on a certain region. For each part, do the following: (i) find the values $f(2i)$ and $f(-2i)$, and (ii) determine the range of f .

- (a) Let the region be the complex plane with the intervals $(-\infty, -1]$ and $[1, \infty)$ of the real axis removed. This plane with two infinite slits is simply connected, and $z^2 - 1$ is never equal to 0 on the region, so there exists an analytic logarithm of $z^2 - 1$ and hence an analytic square root. Let $f(z)$ be the branch of $\sqrt{z^2 - 1}$ on this region such that $f(0) = +i$.
- (b) Let the region be the complex plane with the interval $[-1, 1]$ of the real axis removed. This plane with one finite slit is not simply connected, but there is nonetheless an analytic square root of $z^2 - 1$ on the region. The reason is that a closed loop in this region that circles the branch point 1 must simultaneously circle the branch point -1 , so analytic continuation of a local branch of $\sqrt{z^2 - 1}$ around a closed loop results in two cancelling sign changes. Let $f(z)$ be the branch of $\sqrt{z^2 - 1}$ on this region such that $f(2) = +\sqrt{3}$.

5. Suppose f is analytic on the closed unit disk¹ with Maclaurin series $\sum_{n=0}^{\infty} a_n z^n$, and suppose additionally that $|f(z)| \leq 1$ when $|z| \leq 1$. Then $|a_0| \leq 1$ (since a_0 is a value of the function, namely $a_0 = f(0)$), and in the extreme case that $|a_0| = 1$, the maximum principle implies that $a_n = 0$ when $n \geq 1$ (since in this extreme case f reduces to a constant function).

Extrapolating from the extreme case, one might reasonably expect that when $|a_0|$ is *close* to 1, the other coefficients a_n should be *close* to 0. Your task is to prove the following quantitative formulation² of this expectation: namely, $|a_n| \leq 2(1 - |a_0|)$ when $n \geq 1$. You may wish to follow these hints:

- (i) There is no loss of generality in assuming that a_0 is a non-negative real number. Why?
- (ii) Show that

$$\begin{aligned} -a_n &= \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} [-f(e^{i\theta})] d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} [2 - f(e^{i\theta}) - \overline{f(e^{i\theta})}] d\theta \end{aligned}$$

when $n \geq 1$.

- (iii) Bound the modulus of the integral from above by a simpler integral, observing that the quantity $2 - 2 \operatorname{Re} f(e^{i\theta})$ is non-negative. (Why?)
- (iv) Show that

$$\frac{1}{2\pi} \int_0^{2\pi} [2 - 2 \operatorname{Re} f(e^{i\theta})] d\theta = 2 - 2a_0.$$

- (v) Put the pieces together to get the desired inequality.

¹It is enough to assume that f is analytic on the open unit disk and continuous on the closed disk. This follows by the standard device of working on a slightly smaller disk and then passing to the limit as the radius increases to 1.

²An even stronger inequality is true, but proving it is correspondingly harder.

6. In this problem, you will prove Hadamard's three-circles theorem, a topic that is not covered in the textbook but that is on the syllabus for the complex analysis qualifying examination.

Suppose f is analytic in the closed³ annulus $\{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$. Let $M(r)$ denote $\max\{|f(z)| : |z| = r\}$. The maximum principle implies that $M(r) \leq \max(M(r_1), M(r_2))$ when $r_1 \leq r \leq r_2$.

Hadamard's three-circles theorem says that much more is true: namely, $M(r)$ satisfies a certain convexity property. The property may be written in either of the following two equivalent forms.

$$M(r) \leq M(r_1)^\alpha M(r_2)^{1-\alpha}, \quad \text{where } \alpha = \frac{\log(r_2/r)}{\log(r_2/r_1)}, \quad \text{or}$$

$$\log M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2).$$

In words, the inequality says that $\log M(r)$ is a convex function of $\log r$. That is the language in which Hadamard originally stated the result⁴ (without proof).

[In general, a real-valued function g of a real variable is called a convex function if $g(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$ when $0 \leq t \leq 1$. This condition says geometrically that every chord joining two points of the graph of g lies above the graph of g . Elementary calculus books often refer to convex functions as being "concave up."]

Your task is to supply a proof of the three-circles theorem by examining the function $z^\beta f(z)$, where the real number β is chosen such that $r_1^\beta M(r_1) = r_2^\beta M(r_2)$.

The main technical difficulty that you need to address is this: when β is not an integer, the function $z^\beta f(z)$ is only locally defined (but not globally defined on the annulus). Nonetheless, you should be able to show that the globally defined real-valued function $|z|^\beta |f(z)|$ takes its maximum on the boundary of the annulus. After that, you need only do some routine calculations to produce the desired inequality.

³For the same reason as in the preceding problem, it is enough to assume that f is analytic on the open annulus and continuous on the closed annulus.

⁴J. Hadamard, "Sur les fonctions entières," *Bulletin de la Société Mathématique de France* **24** (1896) 186–187.