Exercise on approximation

The *algebraic* numbers are the complex numbers that are zeroes of nonconstant polynomials having rational coefficients. Examples are $\sqrt[3]{2}$ (a zero of the polynomial $z^3 - 2$) and $-1 + i$ (a zero of the polynomial $z^2 + 2z + 2$). Complex numbers that are not algebraic are called *transcendental*. Since the algebraic numbers form a countable set, "most" numbers are transcendental, but determining whether a specific number is transcendental can be a daunting task. Two especially famous numbers known to be transcendental are e (proved by Hermite[1]) and π (proved by Lindemann[2]).

The transcendental number π is nonetheless a zero of some nonconstant entire function having a Maclaurin series with rational coefficients (the sine function, for instance). What about e? Viewing an entire function as a "polynomial of infinite degree," one might ask which numbers are zeroes of nonconstant entire power series having rational coefficients. The amusing answer, due to Hurwitz,[3] is that *every* complex number has this property; moreover, the entire function can be chosen to have only one zero.[4]

Your task is to show that if c is an arbitrary complex number, then there is an entire function h such that the Maclaurin series of $(z - c)e^{h(z)}$ has rational coefficients (in the sense that the real part and the imaginary part of each coefficient are rational numbers). Moreover, if the number c happens to be real, then the coefficients of the series can be taken to be real rational numbers.

1. Show that if f is holomorphic in a neighborhood of the origin, then f can be expressed as $g - h$, where g is holomorphic in a neighborhood of the origin and has rational Maclaurin coefficients, and h is entire.

2. If $c \neq 0$, then $z - c$ can be written *locally* in a neighborhood of the origin in the form $e^{f(z)}$. What can you deduce from the preceding step?

[1] Charles Hermite, Sur la fonction exponentielle, *Comptes rendus de l'Académie des Sciences* **77** (1873) 18–24, 74–79, 226–233, 285–293.

[2] F. Lindemann, Ueber die Zahl π, *Mathematische Annalen* **20** (1882) 213–225.

[3] A. Hurwitz, Über beständig convergirende Potenzreihen mit rationalen Zahlencoefficienten und vorgeschriebenen Nullstellen, *Acta Mathematica* **14** (1890) 211–215.

[4] There is a standard theorem about zeroes of holomorphic functions that is commonly known as "Hurwitz's theorem" (see the index of the textbook), but that theorem concerns a different topic.