Midterm Examination

Part A

State *three* of the following theorems. When a theorem has multiple versions, you may state any one correct version.

1. Runge's approximation theorem.
2. Mittag-Leffler's theorem about functions with prescribed singularities.
3. Weierstrass's factorization theorem for entire functions.
4. Montel's theorem about normal families.
5. Vitali's theorem about convergence of sequences of holomorphic functions.

Part B

Pick *three* of the following items. For each item, either construct an example satisfying the stated conditions or prove that no example exists (whichever is appropriate).

1. A nonconstant entire function f such that f has no zeroes but f', the derivative, has infinitely many zeroes.
2. A sequence $(p_n(z))$ of polynomials converging uniformly on the closed unit disk to a function that has the unit circle as natural boundary (that is, there is no point of the boundary such that the function extends holomorphically to a neighborhood of the point).
3. A locally bounded family $\mathcal{F}$ of nonconstant, zero-free entire functions such that the family $\{ 1/f : f \in \mathcal{F} \}$ of reciprocals is not locally bounded.
4. A holomorphic function that maps $\{ z \in \mathbb{C} : 0 < |z| < 1 \}$ (the punctured unit disk) surjectively onto $\{ z \in \mathbb{C} : |z| < 1 \}$ (the unit disk).
5. A sequence $(p_n(z))$ of polynomials such that the sequence $(p_n(z)^2)$ of squares converges uniformly on every compact subset of $\mathbb{C}$ to the identity function z.