

NOTES ON THE SKEW-SYMMETRIC REALIZATION PROBLEM IN CONTROL THEORY

ABSTRACT. Some notes on the skew-symmetric realization problem in control theory. These (attempt to) follow the basic outline given by the symmetric case [3].

1. INTRODUCTION

We make the following definitions for our specialization. They are slightly less general than the theory affords since we will be restricting our attention to a subclass of problems in this field.

A *complex realization* of a matrix $G(s) \in \mathbb{C}(s)^{n \times n}$ is a set of matrices (A, B, C) such that $A \in \mathbb{C}^{\delta \times \delta}$, $B \in \mathbb{C}^{\delta \times n}$, and $C \in \mathbb{C}^{n \times \delta}$ and

$$(1.1) \quad G(s) = C(sI - A)^{-1}B.$$

In the literature, the matrix $G(s)$ is called a *complex transfer function*. Given A and B , let H be the matrix

$$[A^{n-1}B \ A^{n-2}B \ \dots \ AB \ B].$$

The pair (A, B) is called *controllable* if the range of H is \mathbb{C}^n . There is a “dual” notion to controllability. We say that (A, B) is *observable* if (A^T, B^T) is controllable. The smallest positive integer δ for which the function $G(s)$ has a realization (A, B, C) as in (1.1) is called the *McMillan degree* of $G(s)$. In this case, such a realization (A, B, C) is called *minimal*. The relationship of minimality and McMillan degree to the previous two notions is the following:

Theorem 1.1. *A realization (A, B, C) is minimal if and only if (C, A) is observable and (A, B) is controllable.*

A transfer function $G(s)$ is called *strictly proper* if $G(\infty) = 0$ (still need to check this).

We begin with a statement of the Kalman realization theorem (note: this is my guess as to the statement since I have not found it yet in [1, 2]).

Theorem 1.2 (Kalman realization theorem). *Let $G(s)$ be an $n \times n$ strictly proper, complex rational transfer function of McMillan degree δ . If (A, B, C) and (A', B', C') are two minimal realizations of $G(s)$, then there exists a unique invertible complex $\delta \times \delta$ matrix such that*

$$(A', B', C') = (SAS^{-1}, SB, CS^{-1}).$$

Notice that the sufficiency of the factorization given in the theorem is clear; it is the necessity that is the heart of the result. Recall that a matrix $S \in \mathbb{C}^{\delta \times \delta}$ is called *complex orthogonal* if $SS^T = I$; the set of such matrices will be denoted by $O(\delta, \mathbb{C})$. We will be focusing our attention here on those $G(s)$ which satisfy $G(s) = -G(-s)^T$, the so-called *skew-symmetric* transfer functions. Correspondingly, there is a notion of a *skew-symmetric realization* of $G(s)$, which is a realization (A, B, C)

such that $A = -A^T$ and $B = C^T$. Notice that if (A, B, C) is a realization of a skew-symmetric transfer function, then so is $(-A^T, C^T, B^T)$.

Proposition 1.3. *Let $G(s) = -G(-s)^T$ be an $n \times n$ strictly proper complex transfer function of McMillan degree δ . Then*

- (1) $G(s)$ has a controllable and observable complex skew-symmetric realization

$$(A, B, C) = (-A^T, C^T, B^T).$$

- (2) If (A, B, C) and (A', B', C') are any two controllable and observable complex skew-symmetric realizations of $G(s)$, then there exists a unique complex orthogonal $S \in O(\delta, \mathbb{C})$ such that

$$(A', B', C') = (SAS^{-1}, SB, CS^{-1}).$$

Proof. We first prove statement (1). Let (A, B, C) be a minimal realization of $G(s)$. By Theorem 1.2, since $(-A^T, C^T, B^T)$ is also a minimal realization, it follows that there is a $\delta \times \delta$ invertible matrix S such that

$$(-A^T, C^T, B^T) = (SAS^{-1}, SB, CS^{-1}).$$

In particular, we have that

$$A = S^{-T}SAS^{-1}S^T, \quad B = S^{-T}SB, \quad \text{and} \quad C = CS^{-1}S^T.$$

Set $R = S^{-T}S$ and notice that R is invertible and satisfies

$$(A, B, C) = (RAR^{-1}, RB, CR^{-1}).$$

By uniqueness in Theorem 1.2, it follows that $R = I$ and thus $S = S^T$.

Given a complex symmetric matrix, it is known to have a factorization XX^T for some complex matrix X [4] (**find exact page**). Moreover, X is uniquely determined up to right factors XT where $T \in O(\delta, \mathbb{C})$. A simple calculation reveals that

$$(X^TAX^{-T}, XB, CX^{-1})$$

is a complex skew-symmetric realization, which proves (1). Statement (2) follows from the uniqueness of X up to right factors T .

Proposition 1.4. *Let $SSRat_{\delta,n}(\mathbb{C})$ denote the set of strictly proper complex skew-symmetric transfer functions of McMillan degree δ . Then $SSRat_{\delta,n}(\mathbb{C})$ is a smooth complex manifold of complex dimension [***INSERT HERE***]. Moreover, it is a nonsingular, irreducible quasi-affine variety.*

This proposition allows us to define a notion of “genericity” by way of Zariski-open subsets of $SSRat_{\delta,n}(\mathbb{C})$.

2. STATEMENT OF PROBLEM

We are now ready to state the problem of interest to us here. Let $G(s) = -G(-s)^T$ be strictly proper and of McMillan degree δ . The complex skew-symmetric eigenvalue assignment problem asks the following:

Problem 2.1. *Given an arbitrary monic polynomial $\phi(s) \in \mathbb{C}[s]$ of degree δ , is there an $n \times n$ complex skew-symmetric matrix F such that the closed loop transfer function*

$$G_F(s) = (I_n - G(s)F)^{-1}G(s)$$

has characteristic polynomial $\phi(s)$?

If for a particular skew-symmetric transfer function, Problem 2.1 has a positive answer, we will say that $G(s)$ is *pole-assignable* in the class of skew-symmetric feedback compensators. Moreover, we say that $G(s)$ is *generically pole-assignable* if the problem is solvable for a generic choice of admissible polynomials $\phi(s)$. \square

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