Section 5.1 Accumulated Change

In chapter 2 we learned how to use the derivative to find the rate of change of a function. Here we see how to go in the other direction. If we know the rate of change, can we find the **accumulated change** of the original function on an interval?

How to find Distance Traveled

The rate of change of distance with repsect to time is **velocity**. The distance traveled by a moving object is given by $d = v \cdot t$. If we know the velocity, can we find the distance traveled?

Example: Suppose a car travels at v=50 mph for 2 hours. What is the total distance traveled?

Example: A car starts moving at time $t = 0$ and goes faster and faster. Its velocity is shown in the following table at 4 second intervals. Estimate how far the car travels during the 12 seconds.

t(seconds)	0	4	8	12
Velocity (ft/sec)	0	4	7	12

Example: A car comes to a stop twelve seconds after the driver applies the brakes. While the brakes are on the following velocities are recorded. Estimate how far the car travels during the 12 seconds.

t(seconds)	0	2	4	6	8	10	12
Velocity (ft/sec)	12	11	7	5	4	2	0

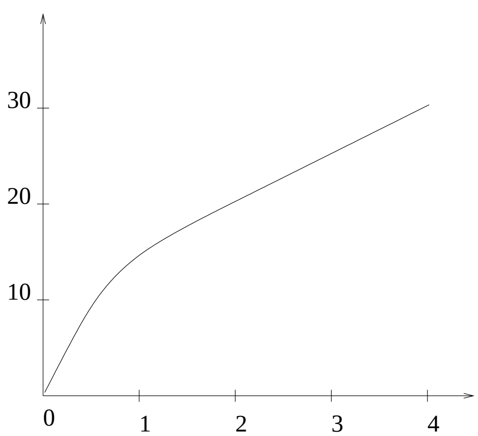
Example: Annual coal production in the US (in quadrillion BTU per year) is given in the following table. Estimate the total amount of coal produced in the US between 1970 and 1990.

Year	1970	1975	1980	1985	1990
Rate of coal production	14.61	14.99	18.60	19.33	22.46

Example: Whisper rips up pages of a magazine. The rate at which she rips up the pages at various times is given in the table below. Determine an upper estimate and a lower estimate of the number of pages Whisper has ripped up during the twenty minute period.

Time in minutes	0	5	10	15	20
Rate in pages per minute	2	5	1	3	8

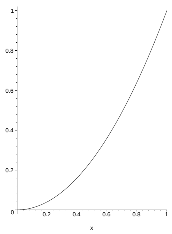
Example: The figure below shows the graph of the velocity v of an object (in meters/sec). Estimate the total distance the object traveled between $t = 0$ and $t = 4$.



Section 5.2 The Definite Integral

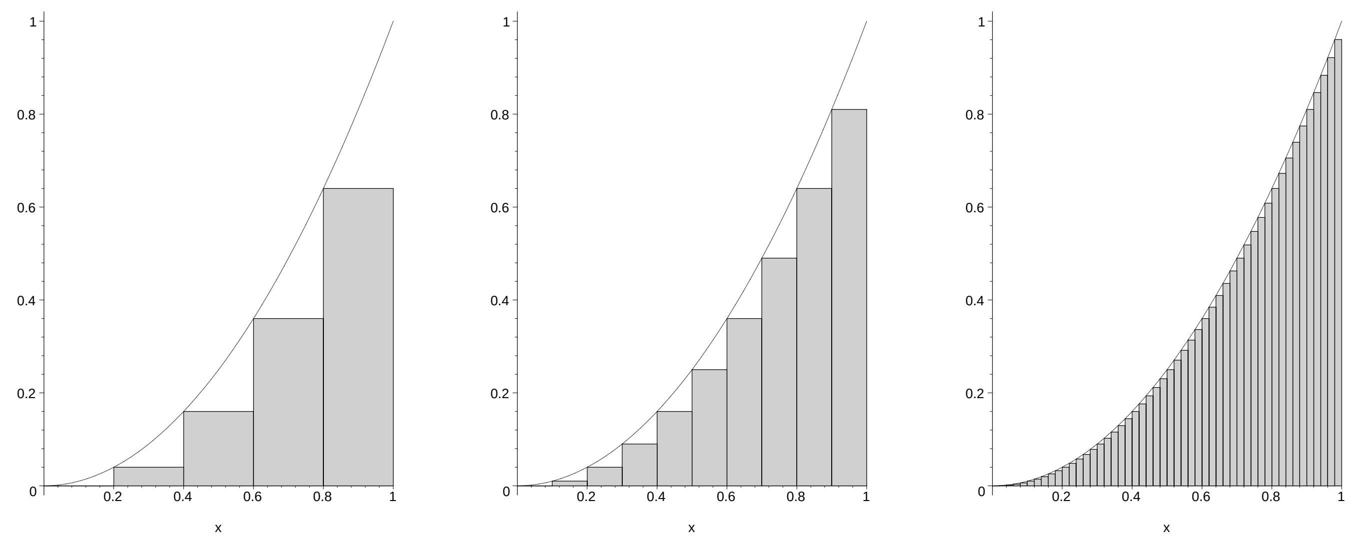
Making our approximations more accurate

Ex An object travels with velocity $v = f(t) = t^2$, where f is in feet per second and t is in seconds. Find estimates, to any desired accuracy, for how far the object traveled during the time-interval [0,1].

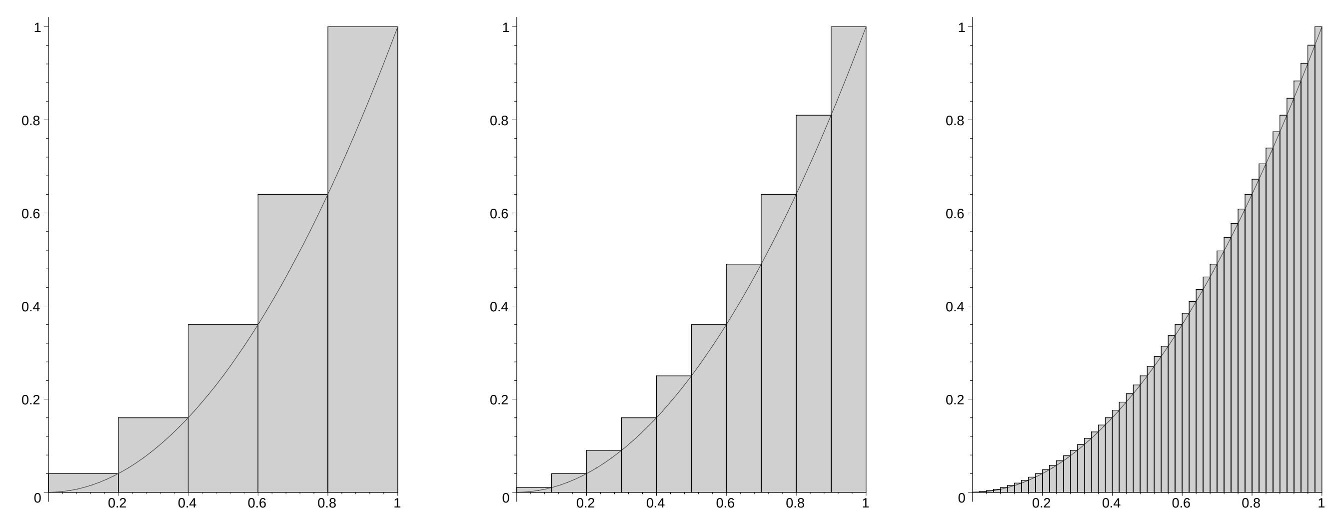


To get better accuracy, we could divide the interval [0,1] into $n = 5$ subintervals of equal length.
Determine an **upper estimate** of the distance traveled.
Determine a **lower estimate** of the distance traveled.
As n (the number of rectangles) gets larger the estimate improves and the area of the shaded rectangles approaches the area under the curve.

Left-hand Sums - These are lower estimates because the function is increasing.



Right-hand Sums -These are upper estimates because the function is increasing.



Conclusion:

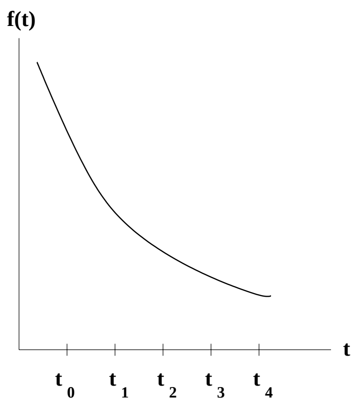
If an object travels at velocity v, during time interval [a,b], then the total distance traveled equals the area between the curve $v = f(t)$ and the t-axis between $t = a$ and $t = b$.

Sigma Notation and Riemann Sums:

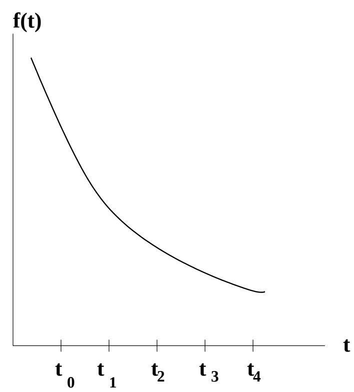
Conclusion: We can use sigma notation to abbreviate the sum of the areas of the rectangles when estimating the area under curve.

$$\text{Right-hand sum} = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{f(t_i) \cdot \Delta t} \qquad \text{Left-hand sum} = \sum_{\mathbf{i=0}}^{\mathbf{n-1}} \mathbf{f(t_i) \cdot \Delta t}$$

The above sums are called Riemann Sums.



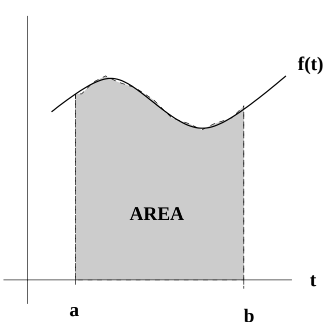
Right−hand sum



Left−hand sum

The Definite Integral

The **definite integral** of f from a to b is defined as follows:



$$\text{AREA}= \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(t)dt} = \lim_{\mathbf{n\to\infty}} (\text{left hand sum}) = \lim_{\mathbf{n\to\infty}} \left(\sum_{\mathbf{i=0}}^{\mathbf{n-1}} \mathbf{f(t_i) \cdot \Delta t}\right)$$

$$\text{AREA}= \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(t)dt} = \lim_{\mathbf{n\to\infty}} (\text{right hand sum}) = \lim_{\mathbf{n\to\infty}} \left(\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{f(t_i) \cdot \Delta t}\right)$$

Example Estimate the given definite integral by computing left-hand and right-hand sums with $n = 2$ and $n = 5$.

$\int_0^2 10(0.85)^x dx$

Now calculate the definite integral using **fnInt**.

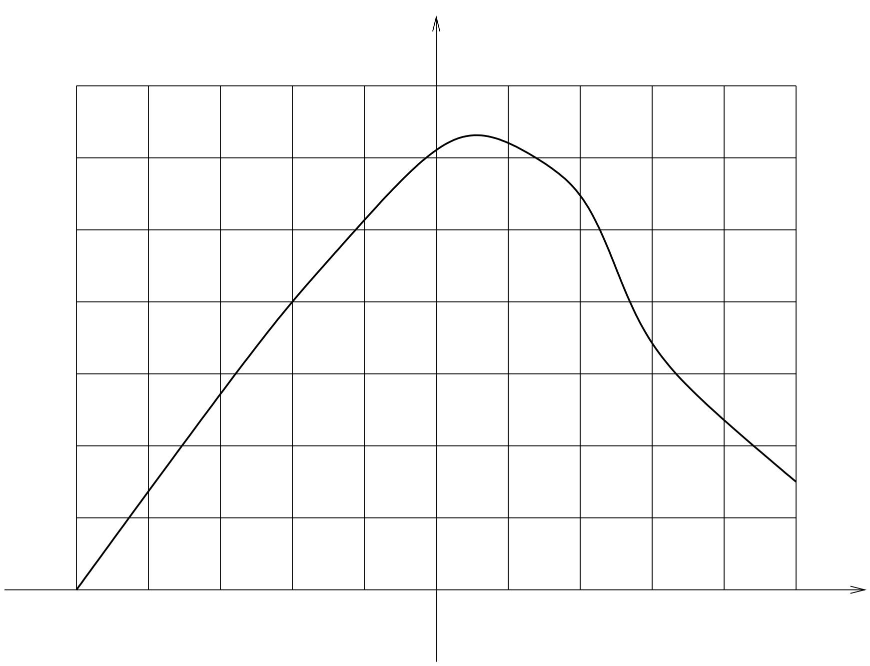
Section 5.3 The Definite Integral as Area

The area of the region, bounded by the curve $f(x)$, the horizontal axis, and the vertical lines $x = a$ and $x = b$ equals the definite integral of f from a to b.

Ex. Find the area under the graph of $f(x) = 3x + 2$ from $x = 1$ to $x = 3$.

Ex. Find the area under the graph of $y = \sqrt{16 - x^2}$ from $x = -4$ to $x = 4$.

Ex. A graph of $f(x)$ is shown below. Use the grid to estimate: $\int_{-2}^{2} f(x)dx$



Positive and Negative
When $f(x)$ is positive for some x-values and negative for others, and $a < c$, the area between $f(x)$ and the x-axis is defined as:

Ex.(a) Find the area between $y = x^3$ and the x-axis from $x = 0$ to $x = 2$.

(b) Find the area between $y = x^3$ and the x-axis from $x = -2$ to $x = 2$.

Ex. Find the area between $y = x^2 - 1$ and the x-axis from $x = -1$ to $x = 2$.

Area Between Two Curves: $y = f(x)$ and $y = g(x)$.
Let $y = f(x)$ and $y = g(x)$ be two continuous functions with $f(x) \geq g(x)$ on [a,b]. Then the area between the graphs of the two curves on [a,b] is given by the definite integral:

Ex. (a) Find the area between $y = x^2 - 1$ and $y = x + 1$.

(b) Find the area between $y = x^2 - 1$ and $y = x + 1$ from $x = -2$ to $x = 2$.

Ex. Find the area of the region between $y = x^2 - 2x + 2$ and $y = 2 + 2x - x^2$

Ex. Find an approximation to the area enclosed between the curves $y = x^5 + x^2 - 3x$ and $y = 0.5x^3$

Section 5.4 Interpretations of the Definite Integral

Ex. If you jump out of an airplane and your parachute fails to open, your downward velocity (in meters per second) t seconds after the jump is approximated by:

$$v(t) = 49(1 - (0.8187)^t)$$

Write an expression for the distance you fall in the first 5 seconds and then calculate the answer.

Ex. A bacteria colony has a population of 14 million bacteria at time $t = 0$. Suppose that the bacteria population is growing at a rate of $f(t) = 2^t$ million bacteria per hour.

(a) Give a definite integral which represents the total change in the bacteria population during the three hours from $t = 0$ to $t = 3$.

(b) Find the population at time $t = 3$.

(c) Find the total change in the population from $t = 3$ to $t = 5$.

Ex. The population of two species of plants have the growth rates shown below. The populations of the two species are equal at time $t = 0$.

Which species has a larger population at the end of 5 years? At the end of 10 years?

Section 5.5 The Fundamental Theorem of Calculus

If $F'(t)$ is continuous for $a \leq t \leq b$, then

$$\int_a^b F'(t)dt = F(b) - F(a)$$

In other words: The definite integral of a derivative gives the total change in quantity of the original function between $t = a$ and $t = b$.

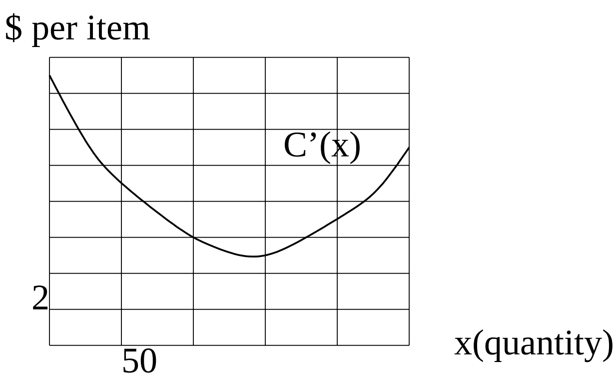
Example The population of a certain city is increasing at the rate given by $P'(t) = 2000e^{t+1}$ where t is time in years measured from the beginning of 1994 when the population was 10,000. Find the population at the end of 1996.

Marginal Cost and Change in Total Cost

The cost function $C(x)$, gives the **total cost** of producing a quantity x of some product.

- $C(x)$ =Fixed costs + Variable Costs
- $C(0)$ is the fixed cost.
- $C'(x)$ is called the **marginal cost** of $C(x)$.
- $\int_0^b C'(x)dx = C(b) - C(0)$= Total variable cost to produce b units
- $\int_a^b C'(x)dx = C(b) - C(a)$= Cost to increase production from a units to b units.

Example A marginal cost curve is given in the following figure. If the fixed cost is \$1000, estimate the total cost of producing 250 items.



Example Below is the graph of $F'(t)$, the rate of change of $F(t)$, an investment, over a 5-month period.

When is the value of the investment increasing and when is it decreasing?

Does the investment increase or decrease in value during the 5 months?

Example If $F(x) = \frac{1}{3}x^3$, it can be shown that $F'(x) = x^2$.
Find $\int_0^1 x^2 dx$ in two ways:

(a) Using a calculator.

(b) Using the Fundamental Theorem of Calculus.

Section 7.1 Constructing Antiderivatives Analytically

$F(x)$ is an **antiderivative** of $f(x)$ if $F'(x) = f(x)$.
Example:

The Indefinite Integral

$$\int f(x)dx = F(x) + C$$