CHAPTER 13: FAIR DIVISION

Matthew and Jennifer must split 6 items between the two of them. There is a car, a piano, a Matisse print, a grandfather clock, an emerald necklace and a diamond ring. How should these items be divided?

A fair division procedure is ***equitable*** if each player believes he or she received the same fractional part of the total value.

A fair division procedure is ***envy-free*** if each player has a strategy that can guarantee him or her a share of whatever is being divided that is, in the eyes of that player, at least as large as that received by any other player, no matter what the other players do.

A fair division procedure is said to be ***Pareto-optimal*** if it produces an allocation of the property that no other allocation can make one player better off without making some other player worse off.

13.1 The Adjusted Winner Procedure

Step 1. Each party distributes 100 points over the items in a way that reflects their relative worth to that party.
Step 2. Each item is initially given to the party that assigns it more points. If there is a tie, the item is not assigned.
Step 3. Each party totals up the number of points it has received and the party that has received the fewest number of points is now given the item that that had a tie.
Step 4. If the number of points each party has is tied, the procedure is complete. If one party has more points, it is named party A and the party with fewer points in named party B.
Step 5. Items are now transferred from party A to party B until the point totals are equal. Fractional transfers are allowed. Transfers are determined using ***point ratios***.

To determine an item's ***point ratio***, find the fraction

$$\frac{\text{A's point value of the item}}{\text{B's point value of the item}}$$

where A is the party with more points. Transfer the item with the lowest point ratio.

Example

Matthew and Jennifer assign point values to the 6 items as shown in the table. How should these items be divided? Will the division found be equitable, envy-free and Pareto-optimal?

Item	Matthew (Step 1)	Jennifer (Step 1)	Step 2	Step 3
Car	20	5	M (55)	70
Piano	35	20	M	
Matisse print	15	15	—	M
Grandfather clock	10	15	J (60)	
Emerald necklace	15	20	J	
Diamond ring	5	25	J	

give up one of these

Step 4. Party A is (A) Matthew (B) Jennifer   Party B is Jennifer

Step 5.

Item	Point Ratio
Car	$20/5 = 4$
Piano	$35/20 = 1.75$
Matisse	$15/15 = 1$ *

Matthew must share this with Jennifer
x of it for Matthew
$1-x$ of it for Jennifer

Matthew's points = Jennifer's points

$$\underset{\text{Car}}{20} + \underset{\text{piano}}{35} + \underset{\text{Mat.}}{15x} = \underset{\text{clock}}{15} + \underset{\text{Neck}}{20} + \underset{\text{ring}}{25} + \underset{\text{Mat.}}{15(1-x)}$$

$$55 + 15x = 60 + 15(1-x) = 60 + 15 - 15x = 75 - 15x$$

$$+15x \qquad\qquad +15x$$

$$55 + 30x = 75$$

$$-55 \qquad -55$$

$$30x = 20 \Rightarrow x = \frac{20}{30} = 2/3$$

Matthew gets the car, the piano and 2/3 of the Matisse.
Jennifer gets the clock, the necklace, the ring and 1/3 of the Matisse

Self check: plug in $x = 2/3 \Rightarrow$ Matthew has 65 points
Jennifer has 65 points

Example

Suppose a labor union and management are trying to resolve a dispute that involves four issues. The points are assigned as shown below. Use the adjusted winner procedure to resolve this conflict.

Issue	Labor (B)		Management (A)	
Base salary	30	divide	50	90
Salary increases	20		40	
Benefits	35		5	
Vacation time	15	50	5	

Point Ratios
50/30 ≈ 1.7
40/20 = 2

Which issue will they have to compromise on (split)?

(A) Base salary

(B) Salary increases

~~(C)~~ Benefits

~~(D)~~ Vacation time

(E) Compromise on more than one issue.

Mg keeps x of Base salary

Mg pts = Lab. points

$$\boxed{\underbrace{40}_{\text{incr}} + \underbrace{50x}_{\text{Base}} = \underbrace{35}_{\text{Ben}} + \underbrace{15}_{\text{V}} + \underbrace{30(1-x)}_{\text{Base}}}$$

$\Rightarrow x = \frac{1}{2}$

$$40 + 50x = 50 + 30 - 30x \Rightarrow 80 - 30x$$
$$+30x \qquad\qquad +30x$$
$$40 + 80x = 80$$
$$-40 \qquad -40$$
$$80x = 40 \Rightarrow x = \frac{40}{80} = \frac{1}{2}$$

Labor gets the Benefits, vacation time and half of their "way" on base salary. Mgt gets their "way" on salary incr and half on base salary

Example

Use the adjusted winner procedure to divide the items below between Harry and Ron.

Item	Harry		Ron	
Cloak	39	80	31	
Radio 96	41		34	
Car	~~16~~		14	35
Tent	4		21	

ratios

cloak 39/31 ≈ 1.258

Radio 41/34 ≈ 1.206 ② transfer this

Car 16/14 ≈ 1.1 ① all to Ron

Tent —

Which of the following is true for Harry?

~~**(A)**~~ ***He has the cloak, the radio and the car***

(B) He has the cloak and the radio. He shares the car with Ron.

(C) He has the cloak and the car. He shares the radio with Ron

(D) He has the cloak and shares the radio with Ron.

13.2 The Knaster Inheritance Procedure

Abe, Betty and Calvin have inherited a house to share equally. How should these be divided?

Step 1. The heirs – independently and simultaneously – submit monetary bids for the object.
Step 2. The high bidder is awarded the object and he or she places all but $1/n$ of his or her bid in a kitty.
Step 3. Each of the other heirs withdraws from the kitty $1/\mathbf{n}$ of his or her bid.
Step 4. The remaining money in the kitty is divided equally

Example
Abe, Betty and Calvin have inherited ~~$300,000 and~~ a house to share equally. Each person writes a bid for the house on a piece of paper. Who gets the house and how much money does each person get? or pay

Step 1. Abe bid $90,000, Betty bid $75,000, and Calvin bid $60,000.

Step 2. *Who gets the house?*
(A) Abe (B) Betty (C) Calvin

Abe put $90000 - \underbrace{30000}_{\frac{1}{3}} = \$60000$ in the kitty

③ Betty thinks $1/3 * 75,000 = \$25,000$ is a third of the value
take from kitty

Calvin thinks $1/3 * 60000 = \$20,000$ is a third of the value
take from kitty

$60000 - 25000 - 20000 = 15,000$
split $15000/3 = 5000$ to each:

Betty gets $25000 + 5000 = \$30,000$
Calvin gets $20000 + 5000 = \$25,000$
Abe gets the house and pays $55,000

13.3 Fair Division and Organ Transplant Policies

13.4 Taking Turns

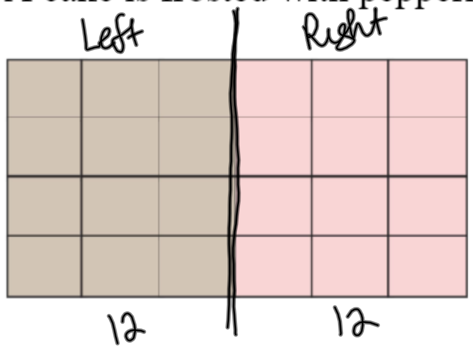
Please read these sections, if you are interested.

13.5 Divide and Chose

With ***divide-and-choose***, one party divides the object into two parts in any way and then the second party chooses one part.

Example

A cake is frosted with peppermint and chocolate frosting as shown.

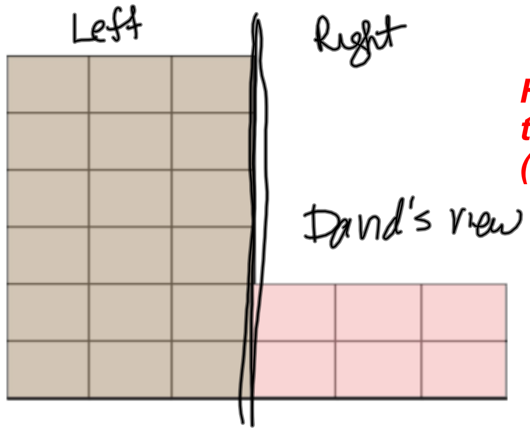


Harry's View

How much is the left half the cake worth to Harry?

(A) 6 (B) 12 (C) 18

Harry and David will split the cake. Harry likes peppermint and chocolate frosting equally, but David likes chocolate much more than peppermint, so to David the cake looks like



How much is the left half the cake worth to David?

(A) 6 (B) 12 (C) 18

If Harry is the divider, how might the results look to David?

13.6 Cake-Division Procedures: Proportionality

A ***cake-division procedure*** for n players is a procedure that the players can use to allocate a cake among them so that each player has a strategy that will guarantee that player a piece with which he or she is "satisfied."

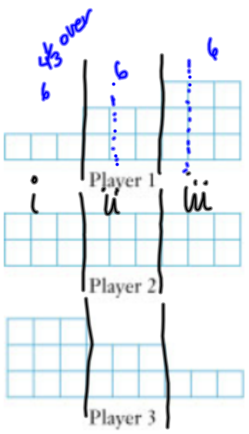
A cake-division procedure for n player is called ***proportional*** if each player's strategy guarantees that player a piece that is worth at least $1/n$ of the whole, in that player's estimation.

The Steinhaus Proportional Procedure (Lone Divider) for Three Players

Step 1. The players (A, B, and C) let player A be the divider.
Step 2. Player A divides the cake into three equal pieces, i, ii, and iii
Step 3. If players B and C each like different pieces, they get those pieces and A gets the remaining piece.
Step 4. If players B and C both want the same piece, they give a not wanted piece to player A. The remaining two pieces are combined and then B divides and C chooses.

This method is proportional but not envy free.

Example
Use the Steinhaus Proportional Procedure to divide the cake below.



Player 2 divides

Player 1 picks iii (value is 9)

$\frac{18}{3} = 6$ worth 6

Player 3 picks i (value of 9)

What if Player 1 divides?

Which piece does player 2 end up with? (A) i (B) ii (C) iii