CHAPTER 14: APPORTIONMENT

14.1 The Apportionment Problem

Exact University needs to create a student government with 24 representatives from 6 groups of students. Here are the enrollment numbers:

	populations, p_i	quotas, q_i
U1	12,000	$\frac{12000}{2000} = 6$
U2	10,000	$\frac{10000}{2000} = 5$
U3	8,000	$\frac{8000}{2000} = 4$
U4	8,000	4
U5	4,000	2
U6	6,000	3
TOTAL	48,000	24

The total population, p, divided by the house size, h, is called the ***standard divisor, s***.

$$s = \frac{p}{h} = \frac{48000}{24} = 2000$$

(for every 2000 people in your group you get a house member)

A group's quota q_i is the group's population, p_i, divided by the standard divisor, s.

$$q_i = \frac{p_i}{s}$$

Messy University has 5 groups of students and needs to elect 12 representatives.

G1	32,000	$\frac{32000}{4083.333} = 7.837$
G2	2,000	$\frac{2000}{4083.333} = 0.490$
G3	5,000	$= 1.224$
G4	6,000	$= 1.469$
G5	4,000	$= .98$
TOTAL	49000	

How many should G2 get?
(A) 0 (B) 1 (C) Not sure

$s = \frac{49000}{12} = 4083.3333$

An ***apportionment problem*** is to round a set of fractions so their sum is maintained at its original value.

The rounding procedure used in an apportionment problem is called an ***apportionment method***.

Notation

- Round q to the nearest integer is $[q]$ and half-integers round up.
- Round q down is $\lfloor q \rfloor$
- Round q up is $\lceil q \rceil$

$\lfloor 2.51 \rfloor = 2$

$\lfloor 2.99 \rfloor = 2$

$[2] = \lfloor 2 \rfloor = \lceil 2 \rceil = 2$

Clicker: What is $\lceil 2.51 \rceil$

(A) 2 (B) 3

The U.S. constitution says the House of Representatives "shall be apportioned among the several states within this union according to their respective Numbers…"

Trivia question: What was the first bill in U.S. history to be vetoed?

14.2 Hamilton Method

1. Round each quota down. There can be a requirement to not round down to zero (House of Representatives) or even one (TAMU Faculty Senate has a minimum of 2 from each college).
2. Calculate the number of seats left to be assigned.
3. Assign the seats to those with the largest fractional parts.

Example
Apply Hamilton's method to Messy U's apportionment.

How many should G2 get?
(A) 0 (B) 1 (C) Not sure

		q	$\lfloor q \rfloor$	$H.\ q$
G1	32,000	7.837	(#2) 7 +1	8
G2	2,000	0.490	(#3) 0 +1	1
G3	5,000	1.224	1	1
G4	6,000	1.469	1	1
G5	4,000	0.98	(#1) 0 +1	1
TOTAL			9	12 ✓

Need 12 : 12-9=3 to give back

Example

A school district received 46 computers to distribute to 5 high schools based on the number of AP statistics students at each school using Hamilton's plan.

$s = \frac{559}{46} = 12.152$

School	#	q	$\lfloor q \rfloor$	$H.\ q$
Alpha	39	$39/12.152 = 3.209$	3	3
Beta	70	5.760	(#1) 5 +1	6
Gamma	18	1.481	(#2) 1 +1	2
Delta	222	18.269	18	18
Epsilon	210	17.281	17	17
TOTAL	559		44	46 ✓

The district has one more to distribute. Reapportion based on 47.

$s = \frac{559}{47} = 11.894$

School	#	q	$\lfloor q \rfloor$	$H\ q$
Alpha	39	$39/11.894 = 3.279$	3	3
Beta	70	$= 5.885$	#1 5 +1	6
Gamma	18	$= 1.513$	1	1 ?
Delta	222	$= 18.665$	#2 18 +1	19
Epsilon	210	$= 17.656$	#3 17 +1	18
TOTAL			44	

How many should Gamma get? ~~(A) 0~~ (B) 1 (C) 2 (D) 3 (E) ???

A ***paradox*** is a statement that is seemingly contradictory or opposed to common sense and yet is perhaps true.

The ***Alabama paradox*** occurs when a state loses a seat as the result of an increase in the house size.

The ***population paradox*** occurs when there is a fixed number of seats and a reapportionment causes a state to lose a seat to another state even though the percent increase in the population of the state that loses the seat is larger than the percent increase of the state that wins the seat.

Consider two numbers, A and B, where $A > B$.

The ***absolute difference*** between the two numbers is $A - B$

The ***relative difference*** between the two numbers is $\frac{A-B}{B} \times 100\%$

Example

Find the absolute and relative differences between the given numbers.

(a) 100 and 101

ABS DIFF $101 - 100 = 1$

Rel Diff $= \frac{101-100}{100} \times 100\% = 1\%$

(b) 1000 and 1001

ABS DIFF $1001 - 1000 = 1$

REL DIFF $= \frac{1001-1000}{1000} \times 100\%$

Relative difference is (A) 1% (B) 0.1% (C) 0.01% (D) None of these

(c) 100 and 200

ABS DIFF: $200 - 100 = 100$

REL DIFF $\frac{200-100}{100} \times 100\% = 100\%$

(d) 600 and 500

ABS DIFF $600 - 500 = 100$

REL DIFF $\frac{600-500}{500} \times 100\% = 20\%$

Example

100 new faculty members will be apportioned using the Hamilton plan to three colleges at a university according to their enrollment in 2000. This will be done again in 2005.

$s = \frac{50105}{100} = 501.05$

College	Students	q 2000	$\lfloor q \rfloor$	H. q
Ag	3,755	$\frac{3755}{501.05} = 7.494$	7 +1	8
Business	36,100	= 72.049	72	72
Science	10,250	= 20.457	20	20
TOTAL	50,105		99	100 ✓

$s = \frac{50{,}300}{100} = 503$

$\frac{3800-3755}{3755}$

College	Students	q 2005	$\lfloor q \rfloor$	H q	diff	pop chg %
Ag	3,800	7.555	7	7	−1	1.198%
Business	36,150	71.869	71 +1	72	0	0.139%
Science	10,350	20.577	20 +1	21	+1	0.976%
TOTAL	50,300		98			

largest % incr

Did a paradox occur?
(A) Yes, the Alabama paradox (B) Yes, the population paradox
(C) A different parasox (D) Not sure

The ***new states paradox*** occurs in a reapportionment in which an increase in the total number of seats causes a shift in the apportionment of existing states. This was discovered in 1907 when Oklahoma joined the union.

Example
A pre-school received 20 picnic tables to distribute to two age level groups, the three-year olds and the four-year olds.

$\frac{51}{9.5} = 5.4$

$s = \frac{190}{20} = 9.5$

Age Group	p	q	$\lfloor q \rfloor$	H. q
3's	71	$\frac{71}{9.5} = 7.474$	7	7
4's	119	$\frac{119}{9.5} = 12.526$ +1	12	13
TOTAL	190		19	20 ✓

Later a two-year old class was added that has 51 students and an additional 5 picnic tables are purchased.

$s = \frac{241}{25} = 9.64$

Age Group	#	q	$\lfloor q \rfloor$	H. q	Before
2's	51	5.29	5	5	—
3's	71	7.365 +1	7	8	7
4's	119	12.344	12	12	13
TOTAL	241		24	25	

What happened to the 3's?
(A) they stayed the same (B) they gained a table (C) they lost a table

14.3 Divisor Method

Clicker Question: clicker on?
(A) Yes

The standard divisor, s, represents the average district population. Apportionment can be done by adjusting the average district population to be a specific value called the ***adjusted divisor, d***.

A ***divisor method*** of apportionment determines each state's apportionment by dividing its population by a common divisor d and rounding the resulting quota. Different divisor methods use different rounding rules.

A ***critical divisor*** is a divisor that will produce a quota for each population that gives a correct total number of seats.

The Jefferson Method

1. Find the standard divisor, s. Then find $q_i = \left\lfloor \frac{p_i}{s} \right\rfloor$
2. If the total number of seats is not correct, find the new divisors that correspond to giving each state one more seat. $d_i = \frac{p_i}{q_i + 1}$
3. Assign a seat to the state with the largest d_i. If the total number of seats is correct, stop. Otherwise repeat step 2.
4. The adjusted divisor d will be the exact value of the last divisor found in step 3.

Example $S = 9.64$

$\frac{71}{7} = 10.14$ $\frac{51}{5} = 10.2$

Age Group	p_i	q	$\lfloor q \rfloor$	d_i	N
2's	51	5.29	5	$\frac{51}{5+1} = 8.5$	5
3's	71	7.365	7	$\frac{71}{7+1} = 8.875$	7
4's	119	12.344	12	$\frac{119}{12+1} = 9.154 = d$	13
			24		

$\frac{119}{12} = 9.9$

Who should get the extra table? (A) 2's (B) 3's (C) 4's

Example

A company will hire 200 new workers to work at one of the four facilities around the state. The new workers will be apportioned using Jefferson's plan according to the current production levels at each facility. The location and production levels are given below.

$d = \frac{18834}{200} = 94.17$

$\frac{12520}{133} = d$

		q	$\lfloor q \rfloor$	d
Abilene	12,520	$\frac{12520}{94.17} = 132.951$	132	$\frac{12520}{132+1} = 94.135$ New divisor
Beaumont	4,555	48.370	48	$\frac{4555}{48+1} = 92.959$
Corpus C.	812	8.623	8	$\frac{812}{8+1} = 90.222$
Dallas	947	10.056	10	$\frac{947}{10+1} = 86.091$
TOTAL	18834		198	

$d = \frac{12520}{133} \approx 94.135$

		q	$\lfloor q \rfloor$	d	N
Abilene	12,520	$\frac{12520}{12520/133} = 133$	133	$\frac{12520}{133+1} = 93.433$ *	134
Beaumont	4,555	$\frac{4555}{94.135} = 48.388$	48	$\frac{4555}{48+1} = 92.959$	48
Corpus C.	812	$\frac{812}{94.135} = 8.626$	8	$= 90.22$	8
Dallas	947	$\frac{947}{94.135} = 10.06$	10	$= 86.091$	10
TOTAL			199		

Quota Rule says that the number assigned to each represented unit must be the standard quota, q_i, rounded up or rounded down.

Balinski and Young found that no apportionment method that satisfies the quota rule is free of paradoxes.

Did anyone violate the quota rule?
(A) Yes, Abilene did (B) Yes, Beaumont did (C) Yes, Corpus Christ did
(D) Yes, Dallas did (E) No quota rule violations

The Adams Method

1. Find the standard divisor, s. Then find $N_i = \lceil q_i \rceil = \left\lceil \frac{p_i}{s} \right\rceil$
2. If the total number of seats is not correct, find the new divisors that correspond to giving each state one fewer seat. $d_i = \frac{p_i}{q_i - 1}$
3. Remove a seat from the state with the smallest d_i. If the total number of seats is correct, stop. Otherwise repeat step 2.
4. The adjusted divisor d will be the exact value of the last divisor found in step 3.

Example
A school district received 47 computers to distribute to 5 high schools based on the number of AP statistics students at each school using Adams' plan.

$s = \frac{558}{47} = 11.8723$

$S = 11.8723$

School	p_i	q_i	$\lceil q_i \rceil$	$d_i = p_i / (\lceil q_i \rceil - 1)$	N
Alpha	39	3.28	4	$\frac{39}{4-1} = 13$	4
Beta	69	5.81	6	$\frac{69}{6-1} = 13.8$	6
Gamma	18	1.52	2	$\frac{18}{2-1} = 18$	2
Delta	222	18.70	19	$\frac{222}{19-1} = 12.33$ -1	18
Epsilon	210	17.69	18	$\frac{210}{18-1} = 12.35$	18
TOTAL	558		49		48

Who loses the first computer?
(A) Alpha (B) Beta (C) Gamma (D) Delta (E) Epsilon

$d = \frac{222}{18} \approx 12.33$ NEW DIVISOR

	quota	$\lceil q \rceil$	d	N
A	$\frac{39}{12.33} = 3.16$	4	$\frac{39}{4-1} = 13$	4
B	$\frac{69}{12.33} \approx 5.59$	6	$\frac{69}{6-1} = 13.8$	6
G	$\frac{18}{12.33} = 1.46$	2	$\frac{18}{2-1} = 18$	2
D	$222/(222/18) = 18$	18	$\frac{222}{18-1} = 13.06$	18
E	$210/12.33 = 17.02$	18	$\frac{210}{18-1} = 12.35$ -1	17
		48		47 ✓

The Webster Method

1. Find the standard divisor, s. Then find $N_i = [q_i] = \left[\frac{p_i}{s}\right]$
2. If the total number of seats is correct, the process is done.
3. If the total number of seats is too few, use a critical divisor of d^+ and the state with the largest critical divisor gets a next seat

$$d_i^+ = \frac{p_i}{N_i + \frac{1}{2}}$$

4. If the total number of seats is too many, use a critical divisor of d^- and the state with the smallest critical divisor loses a seat

$$d_i^- = \frac{p_i}{N_i - \frac{1}{2}}$$

Example

A school district received 46 computers to distribute to 5 high schools based on the number of AP statistics students at each school using Webster's plan.

$$s = \frac{559}{46} = 12.152$$

School	p_i	q_i		$[q_i]$			N
Alpha	39	3.209	3.5	3	$\frac{39}{3+\frac{1}{2}} = 11.143$		3
Beta	70	5.760	5.5	6	$\frac{70}{6+\frac{1}{2}} = 10.769$		6
Gamma	17	1.399	1.5	1	$\frac{17}{1+\frac{1}{2}} = 11.333$		1
Delta	223	18.351	18.5	18	$\frac{223}{18+\frac{1}{2}} = 12.034$	+1	19
Epsilon	210	17.281	17.5	17	$\frac{210}{17+\frac{1}{2}} = 12$		17
TOTAL	559			45			

Who gets the first computer?
(A) Alpha (B) Beta (C) Gamma (D) Delta (E) Epsilon

Geometric Mean

The arithmetic mean of two numbers a and b is given by

$$\bar{x} = \frac{a+b}{2}$$

The ***geometric mean*** of two numbers a and b is given by

$$G(a,b) = \sqrt{ab}$$

Example

Find the arithmetic and geometric means for the following numbers.

(a) 1 and 2. $\bar{x} = \frac{1+2}{2} = 1.5$

$G = \sqrt{1 \times 2} = 1.4142$

How would you round 1.45 using the geometric mean?
(A) 1 (B) 2

(b) 2 and 3 $\bar{x} = \frac{2+3}{2} = 2.5$

$G = \sqrt{2 \times 3} = 2.4495 > 2.46$ if you round by G it would be 3

(c) 10 and 11 $\bar{x} = 10.5$

$G = \sqrt{10 \times 11} = 10.4881$

(d) 50 and 51 $\bar{x} = 50.5$

$G = \sqrt{50 \times 51} = 50.4975\ldots$

$\lceil q \rceil$ up
$\lfloor q \rfloor$ down
$\lceil q \rfloor$ nearest
$\{q\}$ Geo

$\lceil A \rceil$
$\lfloor J \rfloor$
$\lceil W \rfloor$
$\{H\}$

The Hill-Huntington Method

1. Find the standard divisor, s. Then find $q_i = \frac{p_i}{s}$
2. Round each quota q_i up or down by comparing it to $q_i^* = \sqrt{\lfloor q_i \rfloor \times \lceil q_i \rceil}$
3. If the total number of seats is correct, the process is done.
4. If the total number of seats is too few, find the critical divisors

$$d_i^+ = \frac{p_i}{\sqrt{N \times (N+1)}}$$

and the state with the largest critical divisor gets the additional seat. If the house is not yet full, repeat the process

5. If the total number of seats is too many, find the critical divisor

.214 → 1

$$d_i^- = \frac{p_i}{\sqrt{N \times (N-1)}}$$

and the state with the smallest critical divisor loses a seat. If the house is still too full, repeat the process

Example

Apportion based on 46 computers. $s = \frac{558}{46} = 12.13$

School	p	q	q* (G)	
Alpha	39	3.215 3 or 4?	$\sqrt{3 \times 4} = 3.464$	3
Beta	69	5.771		6
Gamma	18	1.484 1 or 2	$\sqrt{1 \times 2} = 1.4142$	2
Delta	222	18.301		18
Epsilon	210	17.312		17
TOTAL	558			46 ✓

Pick Up Quiz ⇒ We will review the Key at the start of class

Example

A town has 3 districts. The North district has a population of 98,000, the East district has a population of 26,000, and the West district has a population of 6,000. The total population is 130,000. Apportion 10 representatives using the following methods:

Clicker Question

(a) Hamilton (b) Jefferson (c) Adams
(d) Webster (e) HH

How many seats does South end up with? (A) 0 (B) 1 (C) 2

(a)	quota	N	
North	98,000/13,000 = 7.538	7+1	8
East	26,000/13,000 = 2	2	2
South	6000/13,000 = 0.461	0	0

★ S = 13,000 ★

(b)	quota	N	d	new d = 12,250	
North	7.538	7	98000/(7+1) = 12,250	98000/12250 = 8	8
East	2	2	26000/(2+1) = 8667	26000/12250 = 2.12	2
South	0.461	0	6000/(0+1) = 6000	6000/12250 = .49	0
		9			10

S = 13,000

(c)	quota	N	d	new divisor d = 14000
North	7.538	8	98000/(8−1) = 14000 (−1)	7
East	2	2	26000/(2−1) = 26000	2
South	0.461	1	6000/(1−1) = BIG#	1
		11		

S = 13,000

HH

(d)	quota	N	(e)	d	N
North	7.538	8	8	$98000/\sqrt{7\times8} = 13096$ (−1)	7
East	2	2	2	$26000/\sqrt{2\times1} = 18385$	2
South	0.461	0	1	$6000/\sqrt{1\times0}$ = BIG# (∞)	1
		10	11		

round up or down, use the geo mean

is it bigger than $G = \sqrt{0\times1} = 0$

0 ——— 1

Apportionment Timeline

- **1787** Constitution drafted by the Constitutional Convention
- **1790** First Census
- **1791** After much debate, Congress approved a bill for a 120 member House and Hamilton's method to apportion seats among the states. Hamilton's method won out over Jefferson's method. Hamilton's method was supported by the Federalists while Jefferson's method was supported by the Republicans.
- **1791** President Washington vetoes the above bill (first veto in US history!).
- **1791** The House, unable to override the veto, passed a new bill for a 105 member House and Jefferson's method to apportion seats among the states.(This method was used until 1840.)
- **1822** Rep. William Lowndes (SC) proposed an apportionment method now known as the Lowndes method. It never passed.
- **1832** John Quincy Adams (former President and, at this time, a representative from Massachusetts) proposes the Adam's method for apportionment. It fails.
- **1832** Senator Daniel Webster (Mass) proposes Webster's method. It fails.
- **1832** Congress passes a bill that retains Jefferson's method but changes the size of the House to 240.
- **1842** Webster's method is adopted and the size of the House is reduced to 223.
- **1852** Rep. Samuel Vinton (Ohio) proposed a bill adopting Hamilton's method with a House size of 233. Congress passes this bill with a change to a House size of 234, a size for which Hamilton's and Webster's methods give the same apportionment.
- **1872** A very confusing year! First the House size was chosen to be 283 so that Hamilton's and Webster's methods would again agree. After much political infighting, 9 more seats were added and the final apportionment did not agree with either method.
- **1876** Rutherford B. Hayes became President based on the botched apportionment of 1872. The electoral college vote was 185 for Hayes and 184 for Tilden. Tilden would have won if the correct apportionment as required by law had been used.
- **1880** The Alabama Paradox surfaced as a major flaw of Hamilton's method.

- **1882** Concerns continued over the flaws in Hamilton's method. Congress passed a bill that kept Hamilton's method but changed the House size to 325 so that Hamilton's method gave the same apportionment as Webster's.
- **1901** The Census Bureau gave Congress tables showing apportionments based on Hamilton's method for all House sizes between 350 and 400.
- **1901** For all House sizes in this range (except for 357) Colorado would get 3 seats. For 357, Colorado would get 2 seats. Rep. Albert Hopkins (IL), chm of the House Committee on Apportionment, submitted a bill using a House size of 357--causing an uproar.
- **1901** Congress defeated Hopkin's bill and instead adopted Webster's method with a House size of 386.
- **1907** Oklahoma joined the union and the New States Paradox was discovered as a result.
- **1911** Webster's method was readopted with a House size of 433. A provision was made to give Arizona and New Mexico each 1 seat if they were admitted to the union.
- **1911** Joseph Hill (chief statistician of the Census Bureau) proposed the Huntington-Hill method.
- **1921** No reapportionment was done after the 1920 census IN DIRECT VIOLATION OF THE CONSTITUTION!
- **1931** Webster's method was adopted with a House size of 435.
- **1941** The Huntington-Hill method was adopted with a House size of 435
- **1990** The U.S. Census Bureau, for only the second time since 1900, allocated Defense Department overseas employees for apportionment purposes. This resulted in Massachusetts losing a seat to Washington. Massachusetts filed suit.
- **1992** Overruling a U. S. district court decision, the U. S. Supreme Court ruled against Massachusetts on technical grounds involving "the separation of powers and the unique constitutional position of the President." (The President is charged with calculating and transmitting the apportionment to Congress.)
- **1992** Montana challenged the constitutionality of the Huntington-Hill method (Montana v. US Dept. of Commerce). The Supreme Court upheld the method. Montana was upset because it lost a seat to Washington based on the results of the 1990 census.