Allometry constants of finite-dimensional spaces: theory and computations

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Abstract

We describe the computations of some intrinsic constants associated to an n-dimensional normed space $\mathcal{V}$, namely the N-th "allometry" constants

$$\kappa_\infty^N(\mathcal{V}) := \inf\left\{\|T\|\cdot\|T'\|, \quad T:\ell_\infty^N\to\mathcal{V},\ T':\mathcal{V}\to\ell_\infty^N,\ TT'=\mathrm{Id}_{\mathcal{V}}\right\}.$$

These are related to Banach–Mazur distances and to several types of projection constants. We also present the results of our computations for some low-dimensional spaces such as sequence spaces, polynomial spaces, and polygonal spaces. An eye is kept on the optimal operators T and T', or equivalently, in the case $N=n$, on the best conditioned bases. In particular, we uncover that the best conditioned bases of quadratic polynomials are not symmetric, and that the Lagrange bases at equidistant nodes are best conditioned in the spaces of trigonometric polynomials of degree at most one and two.

Keywords: Condition numbers, Banach–Mazur distances, projection constants, extreme points, frames.

1 Preamble

Suppose that some computations are to be conducted on the elements of an n-dimensional normed vector space $\mathcal{V}$. In all likelihood, these computations will not to be performed on the vectors themselves, but rather on their coefficients relative to a given basis. In fact, rather than a basis, it can even be advantageous to use a system

$$\underline{v}=(v_1,\ldots,v_N) \qquad \text{with} \quad \mathrm{span}(v_1,\ldots,v_N)=\mathcal{V}.$$

In this finite-dimensional setting, the spanning system $\underline{v}$ may be referred to as a **frame**, especially if the vector space $\mathcal{V}$ coincides with the n-dimensional Euclidean space $\mathcal{E}_n$. In this case, using the vocabulary of frame theory, the **synthesis operator**

$$T: \quad a \quad \in \mathbb{R}^N \mapsto \sum_{j=1}^{N} a_j\, v_j \quad \in \mathcal{E}_n,$$

and its adjoint operator, called the **analysis operator**,

$$T^*: \quad v \quad \in \mathcal{E}_n \mapsto (\langle v, v_j \rangle) \quad \in \mathbb{R}^N,$$

give rise to the **frame operator**, i.e. the symmetric positive definite operator S defined by

$$S := TT^*: \quad v \quad \in \mathcal{E}_n \mapsto \sum_{j=1}^{N} \langle v, v_j \rangle\, v_j \quad \in \mathcal{E}_n.$$

The synthesis operator T admits a canonical right inverse $T^\dagger$, named **pseudo-inverse** of T, which is given by

$$T^\dagger := T^* S^{-1}.$$

The identity $TT^\dagger = \mathrm{Id}_{\mathcal{E}_n}$ translates into the reconstruction formula

$$v = \sum_{j=1}^{N} \langle S^{-1} v, v_j \rangle\, v_j = \sum_{j=1}^{N} \langle v, S^{-1} v_j \rangle\, v_j, \qquad v \in \mathcal{E}_n.$$

Following the reconstruction procedure, the computations involve two steps, namely

$$v \in \mathcal{E}_n \mapsto (\langle v, S^{-1} v_j \rangle) \in \mathbb{R}^N \qquad \text{and} \qquad a \in \mathbb{R}^N \mapsto \sum_{j=1}^{N} a_j v_j \in \mathcal{E}_n.$$

To limit error propagation, we wish to keep these steps stable, hence to control the quantity

$$\sup_{a \in \mathbb{R}^N} \frac{\| \sum_{j=1}^{N} a_j v_j \|}{\|a\|_p} \cdot \sup_{v \in \mathcal{E}_n} \frac{\|(\langle v, S^{-1} v_j \rangle)\|_p}{\|v\|},$$

where the norm on the coefficient vectors is chosen here to be an ℓ_p-norm for some $p \in [1, \infty]$. Observe that the latter quantity can also be written as

$$\|T\| \cdot \|T^\dagger\|, \qquad T : \ell_p^N \to \mathcal{E}_n, \;\; T^\dagger : \mathcal{E}_n \to \ell_p^N.$$

Our purpose is to minimize these quantities, and to highlight the optimal spanning systems, without necessarily restricting the space $\mathcal{V}$ to the Euclidean space $\mathcal{E}_n$. The relevant notions, such as condition number of a spanning system, allometry constant of a space, and so on, are introduced in Section 2. Since the minimization problems at stake are, more often than not, impossible or just very hard to solve explicitly, we base most of our analysis on computational investigations. The optimization procedures used in the paper are described in Section 3. In Section 4, we connect allometry constants with projection constants, thus placing our study in the framework of Banach Space Geometry. We apply our computational approach in the low-dimensional setting in Sections 5 to 9, where we report our findings for the Euclidean space $\mathcal{E}_n$, the sequence space ℓ_p^n, the space $\mathcal{P}_n$ of algebraic polynomials, the space $\mathcal{T}_n$ of trigonometric polynomials, and the polygonal space $\mathcal{G}_m$.

2 Basic concepts

In this section, the main notions — condition number and allometry constant in particular — are introduced, and some elementary properties are presented. After treating the simplest case once and for all, we turn our attention to some preliminary results about the case $p = \infty$, which is the central point of the paper.

2.1 Main definitions

Let us assume that a system $\underline{v} = (v_1, \ldots, v_N)$ spans an n-dimensional normed space $\mathcal{V}$. Then a system $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N)$ of linear functionals on $\mathcal{V}$ is called a **dual system** to $\underline{v}$ if there is a reconstruction formula

$$v = \sum_{j=1}^{N} \lambda_j(v)\, v_j, \qquad v \in \mathcal{V}.$$

When N equals n, this means that $\lambda_j(v_i) = \delta_{i,j}$ for $1 \le i, j \le n$, hence a dual system may also be called a biorthonormal system in this case. As outlined in the introduction, the stability of the two steps involved in the reconstruction formula is measured by the ℓ_p**-condition number** of the spanning system $\underline{v}$ relative to its dual system $\underline{\lambda}$, namely by

$$\kappa_p(\underline{v}|\underline{\lambda}) := \sup_{a \in \ell_p^N} \frac{\left\| \sum_{j=1}^{N} a_j v_j \right\|}{\|a\|_p} \cdot \sup_{v \in \mathcal{V}} \frac{\left\| (\lambda(v_j)) \right\|_p}{\|v\|}.$$

Alternatively, this can be thought of as the condition number of the synthesis operator T relative to its right inverse T', that is

$$\kappa(T|T') := \|T\| \cdot \|T'\|, \qquad \text{where} \quad T : a \in \ell_p^N \mapsto \sum_{j=1}^{N} a_j v_j \in \mathcal{V}, \quad T' : v \in \mathcal{V} \mapsto (\lambda_j(v)) \in \ell_p^N.$$

Since the operator T is surjective, there are many choices for the right inverse T'. We are mainly interested in the ones making the ℓ_p-condition number as small as possible. This gives rise to the **absolute condition number** — or simply condition number — of the synthesis operator T, or of the spanning system $\underline{v}$, as defined by

$$\kappa_p(\underline{v}) = \kappa_p(T) := \inf\left\{\kappa_p(T|T') : \quad T' : \mathcal{V} \to \ell_p^N \text{ right inverse of } T\right\}.$$

The infimum is actually a minimum. We may e.g. argue that the minimization problem

$$\text{minimize} \quad \|T'\| \qquad \text{subject to } TT' = \mathrm{Id}_{\mathcal{V}}$$

can be rephrased as the problem of best approximation to a fixed right inverse from the finite-dimensional space $\left\{U \in L(\mathcal{V}, \ell_p^N) : TU = 0\right\}$. Let us also point out that the minimization problem is convex, so that a local minimum is necessarily a global one.

Our next task consists of minimizing the absolute condition number $\kappa_p(\underline{v})$ over all spanning systems $\underline{v} = (v_1, \ldots, v_N)$. We obtain the N-th order ℓ_p-condition number of the space $\mathcal{V}$, or its **N-th allometry constant over ℓ_p**, which is defined by

$$\kappa_p^N(\mathcal{V}) := \inf\left\{\kappa_p(\underline{v}), \quad \underline{v} = (v_1, \ldots, v_N) \text{ spans } \mathcal{V}\right\}.$$

Once again, the infimum turns out to be a minimum — see Lemma 1. We would ideally like to find, or merely characterize, the minimizers. Since such a task is arduous, we could replace it by the search for nearly optimal systems, in the sense that their condition numbers would behave like the actual minimum asymptotically in n and N. This, however, is not the focus of the present paper. Indeed, our goal is to carry out numerical computations providing exact minima and minimizers for some particular spaces $\mathcal{V}$, and to draw some partial conclusions from there.

The quantity $\kappa_p^N(\mathcal{V})$ being sought shall from now on always be referred to as N-th allometry constant of $\mathcal{V}$ over ℓ_p. Here is a brief explanation for the choice of this term. We point out that a large $\kappa_p^N(\mathcal{V})$ indicates that the elements of $\mathcal{V}$ cannot be represented by elements of ℓ_p^N — the coefficient vectors — with roughly the same norms. Informally, the spaces $\mathcal{V}$ and ℓ_p^N, not being isometric, would be dubbed 'allometric' — '$\alpha\lambda\lambda o\varsigma$' means 'other' in Greek, while '$\iota\sigma o\varsigma$' means 'same'. On the other hand, specifying $p = \infty$, if $\kappa_\infty^N(\mathcal{V})$ takes its smallest possible value, that is if $\kappa_\infty^N(\mathcal{V}) = 1$, then Corollary 8 implies that the spaces $\mathcal{V}$ and ℓ_∞^n are isometric, so that the space $\mathcal{V}$ is not 'allometric' to the superspace ℓ_∞^N of ℓ_∞^n.

2.2 Simple remarks

The following lemmas will be used without any further reference in the rest of the paper.

Lemma 1. There always exists a **best conditioned N-system**, i.e. one can always find a system $\underline{v} = (v_1, \ldots, v_N)$ such that

$$\kappa_p(\underline{v}) = \kappa_p^N(\mathcal{V}).$$

Proof. We shall adopt the 'synthesis operator – right inverse' formalism here. We consider sequences (T_k) and (T'_k) of operators from ℓ_p^N onto $\mathcal{V}$ and from $\mathcal{V}$ into ℓ_p^N, with $T_kT'_k = \mathrm{Id}_\mathcal{V}$, such that $\kappa_p(T_k|T'_k)$ converges to $\kappa_p^N(\mathcal{V})$. Without loss of generality, we may assume that $\|T_k\| = 1$, so that $\|T'_k\|$ is bounded. Hence the sequences (T_k) and (T'_k), or at least some subsequences, converge to some $T : \ell_p^N \to \mathcal{V}$ and $T' : \mathcal{V} \to \ell_p^N$. The identities $T_kT'_k = \mathrm{Id}_\mathcal{V}$ yield $TT' = \mathrm{Id}_\mathcal{V}$, so that T' is a right inverse of T. Besides, since $\kappa_p(T_k|T'_k)$ also converges to $\|T\| \cdot \|T'\| = \kappa_p(T|T')$, we can conclude that $\kappa_p(T|T') = \kappa_p^N(\mathcal{V})$ and that $\kappa_p(T) = \kappa_p^N(\mathcal{V})$. $\square$

Lemma 2. If two n-dimensional spaces $\mathcal{V}$ and $\mathcal{W}$ are isometric, then they have the same allometry constants, i.e

$$\big[\mathcal{V} \cong \mathcal{W}\big] \Longrightarrow \big[\kappa_p^N(\mathcal{V}) = \kappa_p^N(\mathcal{W}) \qquad \text{for any } N \geq n\big].$$

Proof. Let $S : \mathcal{V} \to \mathcal{W}$ be an isometry. Let us pick operators $T : \ell_p^N \to \mathcal{V}$ and $T' : \mathcal{V} \to \ell_p^N$ such that $TT' = \mathrm{Id}_\mathcal{V}$ and $\|T\| \cdot \|T'\| = \kappa_p^N(\mathcal{V})$. Then the operators $ST : \ell_p^N \to \mathcal{W}$ and $T'S^{-1} : \mathcal{W} \to \ell_p^N$ satisfy $(ST)(T'S^{-1}) = \mathrm{Id}_\mathcal{W}$. Hence, we obtain

$$\kappa_p^N(\mathcal{W}) \leq \|ST\| \cdot \|T'S^{-1}\| = \|T\| \cdot \|T'\| = \kappa_p^N(\mathcal{V}).$$

The reverse inequality $\kappa_p^N(\mathcal{V}) \leq \kappa_p^N(\mathcal{W})$ is derived by exchanging the roles of $\mathcal{V}$ and $\mathcal{W}$. $\square$

Lemma 3. Denoting by $\mathcal{V}^*$ the dual space of $\mathcal{V}$ and by p^* the conjugate of p, i.e. $1/p + 1/p^* = 1$, one has

$$\kappa_p^N(\mathcal{V}) = \kappa_{p^*}^N(\mathcal{V}^*).$$

Proof. Let us consider operators $T : \ell_p^N \to \mathcal{V}$ and $T' : \mathcal{V} \to \ell_p^N$ such that $TT' = \mathrm{Id}_{\mathcal{V}}$ and $\|T\| \cdot \|T'\| = \kappa_p^N(\mathcal{V})$. Then the adjoint operators $T^* : \mathcal{V}^* \to (\ell_p^N)^* \cong \ell_{p^*}^N$ and $T'^* : (\ell_p^N)^* \cong \ell_{p^*}^N \to \mathcal{V}^*$ satisfy $T'^*T^* = \mathrm{Id}_{\mathcal{V}^*}$. Hence, we obtain

$$\kappa_{p^*}^N(\mathcal{V}^*) \leq \|T'^*\| \cdot \|T^*\| = \|T'\| \cdot \|T\| = \kappa_p^N(\mathcal{V}).$$

The reverse inequality $\kappa_p^N(\mathcal{V}) \leq \kappa_{p^*}^N(\mathcal{V}^*)$ is derived by exchanging the roles of $\mathcal{V}$ and $\mathcal{V}^*$, and of p and p^*. $\square$

Lemma 4. The sequence $\left(\kappa_p^N(\mathcal{V})\right)_{N \geq n}$ is nonincreasing, bounded below by 1, hence converges.

Proof. The monotonicity of the sequence is the only non-trivial property. To establish it, let us consider a best conditioned N-system $\underline{v}$, together with an optimal dual system $\underline{\lambda}$. We define the $(N+1)$-system $\widetilde{\underline{v}} := (v_1, \ldots, v_N, 0)$ and the dual system $\widetilde{\underline{\lambda}} =: (\lambda_1, \ldots, \lambda_N, 0)$. We simply have to notice that $\kappa_\infty(\widetilde{\underline{v}}|\widetilde{\underline{\lambda}}) \leq \kappa_\infty(\underline{v}|\underline{\lambda})$ to conclude that $\kappa_\infty^{N+1}(\mathcal{V}) \leq \kappa_\infty^N(\mathcal{V})$. $\square$

Let us mention that $\kappa_p^n(\mathcal{V})$, the first term of the sequence $\left(\kappa_p^N(\mathcal{V})\right)_{N \geq n}$, is nothing else than the classical **Banach–Mazur distance** $d(\mathcal{V}, \ell_p^n)$ from $\mathcal{V}$ to ℓ_p^n. Note that it is actually the logarithm of the Banach–Mazur distance that induces a metric on the classes of isometric n-dimensional spaces.

2.3 The most trivial case

In this subsection, we specify $p = 2$ and $\mathcal{V} = \mathcal{E}_n$, the n-dimensional Euclidean space. Because $\kappa_2^n(\mathcal{E}_n) = d(\mathcal{E}_n, \ell_2^n) = 1$ and in view of Lemma 4, we derive

$$\kappa_2^N(\mathcal{E}_n) = 1 \qquad \text{for any } N \geq n.$$

Besides, there is an easy characterization of best ℓ_2-conditioned N-systems. Namely, for a system $\underline{v} = (v_1, \ldots, v_N)$, we claim that

$$[\ \kappa_2(\underline{v}) = 1\ ] \iff [\ \underline{v} \text{ is a tight frame }].$$

We recall that $\underline{v}$ is called a **tight frame** if its canonical dual frame

$$\underline{v}^\dagger := S^{-1}\underline{v}, \qquad S := TT^* \text{ being the frame operator,}$$

reduces — up to scaling — to the frame itself , i.e. if one of the following equivalent conditions holds for some constant c,

$$\Big[\|v\|_2^2 = c\sum_{j=1}^{N}\langle v, v_j\rangle^2, \quad v \in \mathcal{E}_n\Big], \qquad \Big[v = c\sum_{j=1}^{N}\langle v, v_j\rangle\, v_j, \quad v \in \mathcal{E}_n\Big], \qquad \Big[S = c^{-1}\mathrm{Id}_{\mathcal{E}_n}\Big].$$

Note that if the tight frame $\underline{v}$ is in addition **normalized**, i.e. if $\|v_j\| = 1$ for all $j \in [\![1, N]\!]$, then one necessarily has $c = n/N$. Our claim is an immediate consequence of some classical results stated below using the terminology of the paper.

Proposition 5. The absolute ℓ_2-condition number of a frame $\underline{v}$ for $\mathcal{E}_n$ reduces to its **canonical ℓ_2-condition number**, i.e. its ℓ_2-condition number relative to its canonical dual frame. It also equals the square root of the spectral condition number of the frame operator, defined as the ratio of its largest eigenvalue by its smallest one. In short, one has

$$\kappa_2(\underline{v}) = \kappa_2(\underline{v}|\underline{v}^\dagger) = \sqrt{\rho(S)\cdot\rho(S^{-1})}.$$

Proof. For the second equality, we simply write $\kappa_2(\underline{v}|\underline{v}^\dagger) = \|T\|\cdot\|T^\dagger\|$ and make use of

$$\|T\|^2 = \rho(TT^*) = \rho(S), \quad \|T^\dagger\|^2 = \rho(T^{\dagger *}T^\dagger) = \rho((T^*S^{-1})^*\,(T^*S^{-1})) = \rho(S^{-1}TT^*S^{-1}) = \rho(S^{-1}).$$

As for the first equality, we just have to verify that $\|T^\dagger\| \le \|T'\|$ for any right inverse T' of the synthesis operator T. As a matter of fact, if T' is a right inverse of T, we even have $\|T^\dagger v\| \le \|T'v\|$ for any $v \in \mathcal{V}$. Indeed, we may remark that $T^\dagger v$ is the orthogonal projection of $T'v$ on the space $\operatorname{ran} T^\dagger$, since, for any $x \in \mathcal{E}_n$, we have

$$\langle T^\dagger v, T^\dagger x\rangle = \langle TT'v, T^{\dagger *}T^\dagger x\rangle = \langle TT'v, S^{-1}x\rangle = \langle T'v, T^*S^{-1}x\rangle = \langle T'v, T^\dagger x\rangle. \qquad \square$$

2.4 A crucial observation for $p = \infty$

From now on, we shall only consider the case $p = \infty$. As a result, the terms condition number and allometry constant will implicitly be understood to refer to ℓ_∞-condition number and allometry constant over ℓ_∞. Additionally, we will use the notation $\mathrm{Ex}(A)$ to represent the set of extreme points of A, i.e. the elements of the set A that are not strict convex combination of other elements of A. The unit ball of the space $\mathcal{V}$ will furthermore be denoted by $B_{\mathcal{V}}$, its dual unit ball by $B_{\mathcal{V}^*}$, and the unit ball of ℓ_∞^N by B_∞^N.

Let us first give a simplified expression for the condition number of a system $\underline{v} = (v_1, \dots, v_N)$ spanning the space $\mathcal{V}$ relative to a dual system $\underline{\lambda} = (\lambda_1, \dots, \lambda_N)$. The synthesis operator T and its right inverse T' defined by

$$Ta = \sum_{j=1}^{N} a_j v_j, \qquad\qquad T'v = [\lambda_1(v), \dots, \lambda_N(v)]^\top,$$

have norms given by

$$\|T\| = \max_{a \in B_\infty^N} \|Ta\| \qquad = \max_{a \in \mathrm{Ex}(B_\infty^N)} \|Ta\|, \qquad\qquad \text{thus} \quad \|T\| \;= \max_{\varepsilon_j = \pm 1} \Big\| \sum_{j=1}^{N} \varepsilon_j v_j \Big\|,$$

$$\|T'\| = \max_{v \in B_\mathcal{V}} \max_{j \in [\![1,N]\!]} |\lambda_j(v)| \;= \max_{j \in [\![1,N]\!]} \max_{v \in B_\mathcal{V}} |\lambda_j(v)|, \qquad \text{thus} \quad \|T'\| \;= \max_{j \in [\![1,N]\!]} \|\lambda_j\|.$$

Therefore, the condition number of $\underline{v}$ relative to $\underline{\lambda}$ takes the simpler form

$$\kappa_\infty(\underline{v}|\underline{\lambda}) = \max_{\varepsilon_j = \pm 1} \Big\| \sum_{j=1}^{N} \varepsilon_j v_j \Big\| \;\cdot\; \max_{j \in [\![1,N]\!]} \|\lambda_j\|.$$

Using similar arguments, we now indicate how to renormalize the systems $\underline{v}$ and $\underline{\lambda}$ in order for their condition number to be as small as possible. Simply stated, the norms of the dual functionals should all be equal. This is an elementary observation, but it plays an important role in the rest of the paper.

Lemma 6. Consider a system $\underline{v} = (v_1, \dots, v_N)$ that spans the space $\mathcal{V}$, together with a dual system $\underline{\lambda} = (\lambda_1, \dots, \lambda_N)$. One reduces the condition number by setting

$$\underline{v}^\star := (\|\lambda_1\|\, v_1, \dots, \|\lambda_N\|\, v_N) \qquad \text{and} \qquad \underline{\lambda}^\star := (\lambda_1/\|\lambda_1\|, \dots, \lambda_N/\|\lambda_N\|).$$

In other words, one has

$$\kappa_\infty(\underline{v}^\star|\underline{\lambda}^\star) \;\leq\; \kappa_\infty(\underline{v}|\underline{\lambda}).$$

Proof. By construction, we have $\max_{j \in [\![1,N]\!]} \|\lambda_j^\star\| = 1$. Then, with $c := \max_{j \in [\![1,N]\!]} \|\lambda_j\|$, we write

$$\begin{aligned}
\kappa_\infty(\underline{v}^\star|\underline{\lambda}^\star) \;&=\; \max_{\varepsilon_j = \pm 1} \Big\| \sum_{j=1}^{N} \varepsilon_j \, \|\lambda_j\| \, v_j \Big\| \leq \max_{a \in c B_\infty^N} \Big\| \sum_{j=1}^{N} a_j \, v_j \Big\| = \max_{a \in c\, \mathrm{Ex}(B_\infty^N)} \Big\| \sum_{j=1}^{N} a_j \, v_j \Big\| \\
&=\; \max_{\varepsilon_j = \pm 1} \Big\| \sum_{j=1}^{N} c \, \varepsilon_j \; v_j \Big\| = c \cdot \max_{\varepsilon_j = \pm 1} \Big\| \sum_{j=1}^{N} \varepsilon_j v_j \Big\| = \kappa_\infty(\underline{v}|\underline{\lambda}).
\end{aligned}$$

□

3 Description of the computations

We explain in this section the general principles used in our computations of absolute condition numbers and allometry constants. The code, a collection of MATLAB routines, is available on the author's web page, and the reader is encouraged to experiment with it. It relies heavily on MATLAB's optimization toolbox. At this early stage, the computations are not completely trustworthy when the size of the problem ceases to be small, but they can certainly be improved with better suited algorithms implemented in compiled languages. The main ideas, however, remain the same.

We first emphasize that, no matter what space is under consideration, the computations of absolute condition numbers and of the allometry constants follow identical lines. Only the implementations of the norm and of the dual norm differ — see Sections 5 to 9.

For the absolute condition number $\kappa_\infty(\underline{v})$, we calculate, on the one hand, the norm

$$\|T\| = \max_{\varepsilon_j = \pm 1} \Big\| \sum_{j=1}^{N} \varepsilon_j v_j \Big\|, \tag{1}$$

and, on the other hand, we solve the optimization problem

$$\underset{T'}{\text{minimize}} \left[\|T'\| = \max_{j \in [\![1,N]\!]} \|\lambda_j\| \right] \qquad \text{subject to} \quad TT' = \mathrm{Id}_{\mathcal{V}}. \tag{2}$$

We shall fix a suitable basis $\underline{w} = (w_1, \ldots, w_n)$ for $\mathcal{V}$ and represent the system $\underline{v}$ by its matrix in the basis $\underline{w}$, i.e. by the $n \times N$ matrix

$$V = \begin{bmatrix} \mathrm{coef}_{w_1}(v_1) & \ldots & \mathrm{coef}_{w_1}(v_N) \\ \vdots & \ldots & \vdots \\ \mathrm{coef}_{w_n}(v_1) & \ldots & \mathrm{coef}_{w_n}(v_N) \end{bmatrix}.$$

As for the dual system $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N)$, it shall be represented by its values on the basis $\underline{w}$, i.e. by the $n \times N$ matrix

$$U = \begin{bmatrix} \lambda_1(w_1) & \cdots & \lambda_N(w_1) \\ \vdots & \ldots & \vdots \\ \lambda_1(w_n) & \cdots & \lambda_N(w_n) \end{bmatrix}.$$

It is simple to see that the duality condition $v = \sum_{k=1}^{N} \lambda_k(v) v_k$, $v \in \mathcal{V}$, translates into a matrix identity, namely

$$\text{duality conditions:} \qquad VU^\top = I_n.$$

Note that this duality represents a linear equality constraint when the system $\underline{v}$ is fixed, and that the problem (2) is a convex one. We chose to solve it via MATLAB's function `fminimax`, which minimizes the maximum of a finite number of functions, possibly subject to constraints. In fact, the norm $\|\lambda_j\|$ will itself be given as a maximum, so that the number of functions whose maximum is to be minimized typically exceeds N.

We shall also use MATLAB's `fminimax` for the computation of the allometry constant $\kappa_\infty^N(\mathcal{V})$. According to our previous considerations, we simply need to solve the optimization problem

$$\underset{\underline{v},\underline{\lambda}}{\text{minimize}} \max_{\varepsilon_j=\pm 1} \Big\| \sum_{j=1}^{N} \varepsilon_j v_j \Big\| \qquad \text{subject to} \quad VU^\top = I_n \ \text{ and } \ \|\lambda_j\| = 1, j \in [\![1, N]\!].$$

Observe that the additional equality constraints $\|\lambda_j\| = 1$, $j \in [\![1, N]\!]$, are nonlinear, and that the duality condition $VU^\top = I_n$, where both U and V vary, now translates into nonlinear equality constraints, too.

4 Allometry constants and minimal projections

In this section, we point out some connections between allometry constants of a space $\mathcal{V}$ and several projection constants associated to $\mathcal{V}$.

4.1 Allometry constants as projection constants

We briefly recall that a projection P from a superspace $\mathcal{X}$ onto the subspace $\mathcal{V}$ is a linear map acting as the identity on $\mathcal{V}$, and that the **(relative) projection constant** of $\mathcal{V}$ in $\mathcal{X}$ is defined as

$$p(\mathcal{V}, \mathcal{X}) := \inf \big\{ \|P\|,\ P \text{ is a projection from } \mathcal{X} \text{ onto } \mathcal{V} \big\}.$$

The **(absolute) projection constant** of $\mathcal{V}$ is defined as

$$p(\mathcal{V}) := \sup \big\{ p(\mathcal{V}, \mathcal{X}),\ \mathcal{V} \text{ isometrically embedded in } \mathcal{X} \big\}.$$

Note the usual abuse of notation: $p(\mathcal{V}, \mathcal{X})$ stands here for $p(i(\mathcal{V}), \mathcal{X})$, where $i : \mathcal{V} \hookrightarrow \mathcal{X}$ is the isometric embedding. The following connection reveals that projection constants, which are hard to compute, can be approximated by allometry constants, which are computable.

Theorem 7. The sequence of N-th allometry constants converges to the projection constant,

$$\kappa_\infty^N(\mathcal{V}) \underset{N\to\infty}{\longrightarrow} p(\mathcal{V}).$$

Proof. Let $T : \ell_\infty^N \to \mathcal{V}$ be a synthesis operator and $T' : \mathcal{V} \to \ell_\infty^N$ be a right inverse of T. Given a superspace $\mathcal{X}$ of $\mathcal{V}$, we apply the Hahn–Banach theorem componentwise to obtain a linear map $\widetilde{T} : \mathcal{X} \to \ell_\infty^N$ such that $\widetilde{T}_{|\mathcal{V}} = T'$ and $\|\widetilde{T}\| = \|T'\|$. We then define a projection P from $\mathcal{X}$ onto $\mathcal{V}$ by setting $P := T\widetilde{T}$. We observe that

$$p(\mathcal{V}, \mathcal{X}) \leq \|P\| \leq \|T\| \cdot \|\widetilde{T}\| = \|T\| \cdot \|T'\| = \kappa_\infty(T|T').$$

The inequality $p(\mathcal{V}) \leq \kappa_\infty^N(\mathcal{V})$ is obtained by first taking the infimum over T and T', and then the supremum over $\mathcal{X}$. It remains to show that, given any $\varepsilon > 0$, there is an integer N for which $\kappa_\infty^N(\mathcal{V}) \leq (1+\varepsilon) \cdot p(\mathcal{V})$. A classical result states that $\mathcal{V}$ can be embedded into some N-dimensional space $\mathcal{W}$ close to ℓ_∞^N in the sense that $d(\mathcal{W}, \ell_\infty^N) \leq 1+\varepsilon$. Let us consider a minimal projection P from $\mathcal{W}$ onto $\mathcal{V}$ and an isomorphism S from ℓ_∞^N onto $\mathcal{W}$ such that $\|S\| \cdot \|S^{-1}\| \leq 1+\varepsilon$. We then define a synthesis operator T from ℓ_∞^N onto $\mathcal{V}$ by setting $T := PS$, and introduce the operator T' from $\mathcal{V}$ into ℓ_∞^N given by $T' := S^{-1}{}_{|\mathcal{V}}$. It is clear that T' is a right inverse of T, and our claim follows from

$$\kappa_\infty^N(\mathcal{V}) \leq \|T\| \cdot \|T'\| = \|PS\| \cdot \|S^{-1}{}_{|\mathcal{V}}\| \leq \|P\| \cdot \|S\| \cdot \|S^{-1}\| = p(\mathcal{V}, \mathcal{W}) \cdot \|S\| \cdot \|S^{-1}\| \leq p(\mathcal{V}) \cdot (1+\varepsilon).$$

□

Corollary 8. If $\kappa_\infty^N(\mathcal{V}) = 1$ holds for some $N \geq n$, then the space $\mathcal{V}$ is isometric to ℓ_∞^n.

Proof. According to the inequality $p(\mathcal{V}) \leq \kappa_\infty^N(\mathcal{V})$, the equality $\kappa_\infty^N(\mathcal{V}) = 1$ implies the equality $p(\mathcal{V}) = 1$. The latter is known to imply $\mathcal{V} \cong \ell_\infty^n$. □

Theorem 9. If the dual unit ball of the space $\mathcal{V}$ has exactly $2N$ extreme points, then

$$\kappa_\infty^N(\mathcal{V}) = p(\mathcal{V}).$$

Proof. If the dual unit ball $B_{\mathcal{V}^*}$ has exactly $2N$ extreme points, then the isometric embedding $\mathcal{V} \hookrightarrow \mathcal{C}(\mathrm{Ex}(B_{\mathcal{V}^*}))$ simply reads $\mathcal{V} \hookrightarrow \ell_\infty^N$. But then we can take $\varepsilon = 0$ in the proof of Theorem 7 and conclude that $p(\mathcal{V}) = \kappa_\infty^N(\mathcal{V})$. □

4.2 Discrete and pseudodiscrete projections

We shall now lift the restriction imposed in Theorem 9 on the dual unit ball to reveal other connections between N-th allometry constants and projection constants of certain types. We suppose that $\mathcal{V}$ is a subspace of some $\mathcal{C}(K)$, K compact Hausdorff space. This can always be done by taking K as the unit ball $B_{\mathcal{V}^*}$ of the dual space $\mathcal{V}^*$, or as the weak*-closure $\mathrm{cl}[\mathrm{Ex}(B_{\mathcal{V}^*})]$ of the set of extreme points of $B_{\mathcal{V}^*}$, or even as the interval $[-1,1]$ in this finite-dimensional setting. We say that a projection P from $\mathcal{C}(K)$ onto $\mathcal{V}$ is N**-discrete** if it can be written as

(3) $$Pf = \sum_{j=1}^{N} f(t_j)\, v_j \qquad \text{for distinct points } t_1,\ldots,t_N \in K \text{ and vectors } v_1,\ldots,v_N \in \mathcal{V}.$$

We say that it is N**-pseudodiscrete** if it can be written as

(4) $$Pf = \sum_{j=1}^{N} \widetilde{\lambda}_j(f)\, v_j \qquad \text{for linear functionals } \widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_N \text{ on } \mathcal{C}(K) \text{ having disjoint carriers and vectors } v_1,\ldots,v_N \in \mathcal{V}.$$

Recall that the **carrier** $\mathrm{car}(\widetilde{\lambda})$ of a linear functional $\widetilde{\lambda}$ on $\mathcal{C}(K)$ is the smallest closed subset C of K such that $f_{|C} = 0$ implies $\widetilde{\lambda}(f) = 0$. Discrete projections have been studied in [11, 12], where they were called finitely carried projections. The term discrete projection is nonetheless standard, unlike the term pseudodiscrete projection. In fact, when N equals n, an n-pseudodiscrete projection is usually [2, 4, 7] referred to as a **generalized interpolating projection**. In any case, the interesting feature of these projections is the simple expression of their norms.

Lemma 10. If P is an N-pseudodiscrete projection from $\mathcal{C}(K)$ onto $\mathcal{V}$ of the form (4), then

$$\|P\| = \Big\| \sum_{j=1}^{N} \|\widetilde{\lambda}_j\|\, |v_j| \Big\| = \max_{\varepsilon_j = \pm 1} \Big\| \sum_{j=1}^{N} \varepsilon_j \|\widetilde{\lambda}_i\|\, v_j \Big\|.$$

Proof. The following is an adaptation of the classical proof [2] for $N = n$. For $f \in \mathcal{C}(K)$ and $x \in K$, we write

$$|P(f)(x)| = \Big| \sum_{j=1}^{N} \widetilde{\lambda}_j(f) v_j(x) \Big| \le \sum_{j=1}^{N} |\widetilde{\lambda}_j(f)||v_j(x)| \le \sum_{j=1}^{N} \|\widetilde{\lambda}_j\||v_j(x)| \cdot \|f\| \le \Big\| \sum_{j=1}^{N} \|\widetilde{\lambda}_j\|\, |v_j| \Big\| \cdot \|f\|,$$

which yields the inequality

$$\|P\| \le \Big\| \sum_{j=1}^N \|\widetilde{\lambda}_j\| \, |v_j| \Big\|.$$

To establish the reverse inequality, we consider $\varepsilon > 0$ arbitrary small. Let $x^* \in K$ be such that $\sum_{j=1}^N \|\widetilde{\lambda}_j\| |v_j(x^*)| = \| \sum_{j=1}^N \|\widetilde{\lambda}_j\| \, |v_j| \|$. We note first that for a linear functional $\widetilde{\lambda}$ on $\mathcal{C}(K)$, there exists a unique $\widehat{\lambda} \in \mathcal{C}(\operatorname{car} \widetilde{\lambda})$ such that $\widetilde{\lambda}(f) = \widehat{\lambda}(f_{|\operatorname{car} \widetilde{\lambda}})$, $f \in \mathcal{C}(K)$, and that it satisfies $\|\widehat{\lambda}\| = \|\widetilde{\lambda}\|$. Writing $K_j := \operatorname{car} \widetilde{\lambda}_j$, we choose $f_j \in \mathcal{C}(K_j)$, $\|f_j\| = 1$, such that

$$\operatorname{sgn}[v_j(x^*)] \cdot \widehat{\lambda}_j(f_j) \ge (1-\varepsilon) \|\widehat{\lambda}_j\|.$$

By Tietze's theorem, because of the disjointness of the carriers K_j, there exists a function $f \in \mathcal{C}(K)$, $\|f\| = 1$, such that $f_{|K_j} = f_j$ for any $j \in [\![1, N]\!]$. We then get

$$\begin{aligned} \|P\| \ge \|P(f)\| \ge P(f)(x^*) &= \sum_{j=1}^N \widetilde{\lambda}_j(f) v_j(x^*) = \sum_{j=1}^N \widehat{\lambda}_j(f_j) v_j(x^*) \\ &\ge \sum_{j=1}^N (1-\varepsilon)\|\widehat{\lambda}_j\| \, |v_j(x^*)| = (1-\varepsilon) \cdot \sum_{j=1}^N \|\widetilde{\lambda}_j\| \, |v_j(x^*)| = (1-\varepsilon) \cdot \Big\| \sum_{j=1}^N \|\widetilde{\lambda}_j\| \, |v_j| \Big\|. \end{aligned}$$

Taking the limit as ε tends to zero, we obtain the required inequality. □

We now define the N**-discrete projection constant** and N**-pseudodiscrete projection constant** of $\mathcal{V}$ in $\mathcal{C}(K)$ in a straightforward manner by setting

$$\begin{aligned} p^N_{\mathrm{disc}}(\mathcal{V}, \mathcal{C}(K)) &:= \inf \big\{ \|P\|, \ P \text{ is an } N\text{-discrete projection from } \mathcal{C}(K) \text{ onto } \mathcal{V} \big\}, \\ p^N_{\mathrm{pdisc}}(\mathcal{V}, \mathcal{C}(K)) &:= \inf \big\{ \|P\|, \ P \text{ is an } N\text{-pseudodiscrete projection from } \mathcal{C}(K) \text{ onto } \mathcal{V} \big\}. \end{aligned}$$

4.3 Allometry constants as discrete projection constants

Unlike the projection constant of $\mathcal{V}$ in $\mathcal{C}(K)$, the N-discrete projection constant $p^N_{\mathrm{disc}}(\mathcal{V}, \mathcal{C}(K))$ is not independent of K — see [8]. Therefore, we shall introduce a discrete projection constant that is intrinsic to the space $\mathcal{V}$. We formulate an elementary lemma and outline a second connection between allometry constants and projection constants as a preliminary step.

Lemma 11. If the space $\mathcal{V}$ is (isometric to) a subspace of some $\mathcal{C}(K)$, one has

$$\kappa^N_\infty(\mathcal{V}) \ \le \ p^N_{\mathrm{pdisc}}(\mathcal{V}, \mathcal{C}(K)) \ \le \ p^N_{\mathrm{disc}}(\mathcal{V}, \mathcal{C}(K)).$$

Proof. The second inequality is straightforward, because any N-discrete projection is also an N-pseudodiscrete projection. To establish the first inequality, we consider an N-pseudodiscrete projection P of the form (4). The systems $\underline{v} := (v_1, \ldots, v_N)$ and $\underline{\lambda} := (\widetilde{\lambda}_{1|\mathcal{V}}, \ldots, \widetilde{\lambda}_{N|\mathcal{V}})$ being dual, we can form the renormalized dual systems $\underline{v}^\star$ and $\underline{\lambda}^\star$ introduced in Lemma 6. Then, in view of $\|\lambda_j\| = \|\widetilde{\lambda}_{j|\mathcal{V}}\| \le \|\widetilde{\lambda}_j\|$, we get

$$\|P\| = \Big\| \sum_{j=1}^N \|\widetilde{\lambda}_j\| |v_j| \Big\| \ge \Big\| \sum_{j=1}^N \|\lambda_j\| |v_j| \Big\| = \kappa_\infty(\underline{v}^\star | \underline{\lambda}^\star) \ge \kappa_\infty^N(\mathcal{V}).$$

The inequality $p^N_{\rm pdisc}(\mathcal{V}, \mathcal{C}(K)) \ge \kappa^N_\infty(\mathcal{V})$ is finally derived by taking the infimum over P. $\square$

Theorem 12. The N-th allometry constant of the space $\mathcal{V}$ equals its N-discrete projection constant relative to $\mathcal{C}(B_{V^*})$, that is

$$\kappa^N_\infty(\mathcal{V}) = p^N_{\rm disc}(\mathcal{V}, \mathcal{C}(B_{\mathcal{V}^*})).$$

Proof. In view of Lemma 11, we just have to establish that

$$p^N_{\rm disc}(\mathcal{V}, \mathcal{C}(B_{\mathcal{V}^*})) \le \kappa^N_\infty(\mathcal{V}).$$

Let us consider a best conditioned N-system $\underline{v}$, together with an optimal dual system $\underline{\lambda}$. According to Lemma 6, we may assume that $\|\lambda_j\| = 1$, hence $\lambda_j \in B_{\mathcal{V}^*}$. The expression $Pf := \sum_{j=1}^N f(\lambda_j) v_j$ then defines an N-discrete projection from $\mathcal{C}(B_{\mathcal{V}^*})$ onto $\mathcal{V}$. As such, its norm is given by

$$\|P\| = \Big\| \sum_{j=1}^N |v_j| \Big\| = \max_{\varepsilon_j = \pm 1} \Big\| \sum_{j=1}^N \varepsilon_j v_j \Big\| = \kappa_\infty(\underline{v} | \underline{\lambda}).$$

The expected inequality is obtained from $p^N_{\rm disc}(\mathcal{V}, \mathcal{C}(B_{\mathcal{V}^*})) \le \|P\|$ and $\kappa_\infty(\underline{v}|\underline{\lambda}) = \kappa^N_\infty(\mathcal{V})$. $\square$

We may now introduce the N-discrete projection constant by mimicking the definition of the absolute projection constant.

Theorem 13. The **(absolute) N-discrete projection constant** of the space $\mathcal{V}$, namely

$$p^N_{\rm disc}(\mathcal{V}) := \sup \big\{ p_{\rm disc}(\mathcal{V}, \mathcal{C}(K)),\ \mathcal{V} \hookrightarrow \mathcal{C}(K) \big\},$$

is achieved when K is the weak*-closure $E_{\mathcal{V}^*} := {\rm cl}[{\rm Ex}(B_{\mathcal{V}^*})]$ of the set of extreme points of the dual unit ball $B_{\mathcal{V}^*}$. In short, one has

$$p^N_{\rm disc}(\mathcal{V}) = p^N_{\rm disc}(\mathcal{V}, \mathcal{C}(E_{\mathcal{V}^*})).$$

Proof. We need to prove that, whenever $\mathcal{V}$ is (isometric to) a subspace of some $\mathcal{C}(K)$, we have

$$p_{\rm disc}^N(\mathcal{V}, \mathcal{C}(K)) \le p_{\rm disc}^N(\mathcal{V}, \mathcal{C}(E_{\mathcal{V}^*})).$$

Let the expression $Pf = \sum_{j=1}^N f(\lambda_j) v_j$ define an N-discrete projection from $\mathcal{C}(E_{\mathcal{V}^*})$ onto $\mathcal{V}$. The linear functionals λ_j can be associated to points $t_j \in K$, since

$$\mathrm{cl}[\mathrm{Ex}(B_{\mathcal{V}^*})] \subseteq \big\{ \widetilde{\lambda}_{|\mathcal{V}},\ \widetilde{\lambda} \in \mathrm{Ex}(B_{\mathcal{C}(K)^*}) \big\} = \big\{ \pm \delta_{t|\mathcal{V}}, t \in K \big\},$$

where the linear functional δ_t represent the evaluation at the point t. Thus, the expression $\widetilde{P} f = \sum_{j=1}^N f(t_j) v_j$ defines an N-discrete projection form $\mathcal{C}(K)$ onto $\mathcal{V}$. We then derive

$$p_{\rm disc}^N(\mathcal{V}, \mathcal{C}(K)) \le \|\widetilde{P}\| = \max_{\varepsilon_j = \pm 1} \Big\| \sum_{j=1}^N \varepsilon_j\, v_j \Big\| = \|P\|.$$

The required result is now obtained by taking the infimum over P. □

Corollary 14. If the space $\mathcal{V}$ is smooth, then the N-th allometry constant of $\mathcal{V}$ equals its N-discrete projection constant relative to any superspace $\mathcal{C}(K)$, i.e.

$$\kappa_\infty^N(\mathcal{V}) = p_{\rm disc}^N(\mathcal{V}).$$

Proof. The space $\mathcal{V}$ being smooth, we deduce that the dual space $\mathcal{V}^*$ is strictly convex. The latter means that $E_{\mathcal{V}^*} = B_{\mathcal{V}^*}$. We just have to combine Theorems 12 and 13. □

4.4 Allometry constants as pseudodiscrete projection constants

The main drawback of Theorem 12 is that the space $\mathcal{C}(B_{\mathcal{V}^*})$ is not conveniently described. It can in fact be replaced by the space $\mathcal{C}[-1,1]$ if the allometry constants are this time to be identified with pseudodiscrete projection constants. The following generalizes [7, Theorem 2].

Theorem 15. The N-th allometry constant of a space $\mathcal{V}$ equals its N-pseudodiscrete projection constant relative to $\mathcal{C}[-1,1]$, that is

$$\kappa_\infty^N(\mathcal{V}) = p_{\rm pdisc}^N(\mathcal{V}, \mathcal{C}[-1,1]).$$

Proof. Lemma 11 yields the inequality $\kappa_\infty^N(\mathcal{V}) \le p_{\mathrm{pdisc}}^N(\mathcal{V},\mathcal{C}[-1,1])$. We now concentrate on the reverse inequality. Let us consider a best conditioned N-systems $\underline{v} = (v_1,\dots,v_N)$, together with an optimal dual system $\underline{\lambda} = (\lambda_1,\dots,\lambda_N)$. Each linear functional λ_j admits a norm-preserving extension to $\mathcal{C}[-1,1]$ of the form — see [14, Theorem 2.13] —

$$\widetilde{\lambda}_j(f) = \sum_{i=1}^n \alpha_{i,j}\, f(t_{i,j}), \qquad t_{i,j}\in[-1,1], \qquad \sum_{i=1}^n |\alpha_{i,j}| = \|\lambda_j\|. \tag{5}$$

There exist sequences of points $(t_{i,j}^k)_k$ converging to $t_{i,j}$ for which the $t_{i,j}^k$ are all distinct when k is fixed. We then consider the linear functional defined on $\mathcal{C}[-1,1]$ by

$$\widetilde{\lambda}_j^k(f) := \sum_{i=1}^n \alpha_{i,j}\, f(t_{i,j}^k), \qquad \text{so that} \quad \|\widetilde{\lambda}_j^k\| = \sum_{i=1}^n |\alpha_{i,j}| = \|\lambda_j\|.$$

We introduce the linear functionals on $\mathcal{V}$ defined by $\lambda_j^k := \widetilde{\lambda}^k_{j\,|\mathcal{V}}$. As we have done in Section 3, we fix a basis $\underline{w} = (w_1\dots,w_n)$ of $\mathcal{V}$ and we define $n\times N$ matrices U^k and U by

$$U_{i,j}^k = \lambda_j^k(w_i), \quad U_{i,j} = \lambda_j(w_i), \qquad i\in[\![1,n]\!], \quad j\in[\![1,N]\!].$$

Thus, if V represents the $n\times N$ matrix of the system $\underline{v}$ in the basis $\underline{w}$, the duality condition between $\underline{v}$ and $\underline{\lambda}$ simply reads

$$VU^\top = I_n.$$

Note that $VU^{k\top}$ is invertible for k large enough, since $VU^{k\top} \longrightarrow VU^\top = I_n$. We then define

$$V^k := \big(VU^{k\top}\big)^{-1}V.$$

It is readily observed that

$$V^kU^{k\top} = I_n \quad \text{and} \quad V^k \longrightarrow V.$$

This means that the system $\underline{v}^k = (v_1^k,\dots,v_N^k)$, whose matrix in the basis $\underline{w}$ is the matrix V^k, is dual to the system $\lambda^k = (\lambda_1^k,\dots,\lambda_N^k)$ and that it converges to the system $\underline{v} = (v_1,\dots,v_N)$. Therefore, the expression $P^kf := \sum_{j=1}^N \widetilde{\lambda}_i^k(f)\, v_i^k$ defines an N-pseudodiscrete projection from $\mathcal{C}[-1,1]$ onto $\mathcal{V}$ and we obtain

$$\begin{aligned} p_{\mathrm{pdisc}}^N(\mathcal{V},\ \mathcal{C}[-1,1]) &\le \lim_{k\to\infty}\|P^k\| = \lim_{k\to\infty}\Big\|\sum_{j=1}^N \|\widetilde{\lambda}_j^k\|\,|v_j^k|\Big\| = \lim_{k\to\infty}\Big\|\sum_{j=1}^N \|\lambda_j\|\,|v_j^k|\Big\| \\ &= \Big\|\sum_{j=1}^N \|\lambda_j\|\,|v_j|\Big\| \le \max_{j\in[\![1,N]\!]}\|\lambda_j\|\cdot\Big\|\sum_{j=1}^N |v_j|\Big\| = \kappa_\infty(\underline{v}|\underline{\lambda}) = \kappa_\infty^N(\mathcal{V}). \end{aligned}$$

The reverse inequality is therefore established, which concludes the proof. $\square$

5 The Euclidean space

The rest of the paper — except the concluding section — is devoted to the computational analysis of some special spaces $\mathcal{V}$. We begin with the n-dimensional Euclidean space $\mathcal{E}_n$. Note that the frame terminology is used, so that spanning systems and dual systems of linear functionals are called here frames and dual frames.

5.1 Optimal dual frames

In this subsection, we present a condition under which canonical dual frames are optimal.

Proposition 16. If the canonical dual frame $\underline{v}^\dagger$ of a frame $\underline{v} = (v_1, \ldots, v_N)$ is normalized, then it is the unique optimal dual frame, i.e.

$$\big[\|v_j^\dagger\| \text{ independent of } j \in [\![1, N]\!]\big] \Longrightarrow \big[\kappa_\infty(\underline{v}|\underline{v}^\dagger) = \kappa_\infty(\underline{v})\big].$$

Proof. We need to prove that, for any dual frame $\underline{u}$ of $\underline{v}$, there holds

$$\max_{j\in[\![1,N]\!]} \|v_j^\dagger\| \le \max_{j\in[\![1,N]\!]} \|u_j\|.$$

Because of the normalization, it is enough to establish that

$$\sum_{j=1}^N \|v_j^\dagger\|^2 \le \sum_{j=1}^N \|u_j\|^2.$$

Observe that the latter quantity is the squared Frobenius norm,

$$|U|^2 := \mathrm{Tr}(UU^\top),$$

of the $n\times N$ of the matrix $U := \Big[u_1\Big| \;\cdots\; \Big|u_N\Big]$ whose columns are the vectors u_j. Note also that the duality condition between $\underline{v}$ and $\underline{u}$ translates into $VU^\top = I_n$, where $V := \Big[v_1\Big| \;\cdots\; \Big|v_N\Big]$ is the $n \times N$ matrix whose columns are the vectors v_j. Hence, we need to show that the $n \times N$ matrix $V^\dagger := \Big[v_1^\dagger\Big| \;\cdots\; \Big|v_N^\dagger\Big]$ is the unique solution of the minimization problem

$$\text{minimize } \; |U|^2 \qquad \text{subject to} \quad VU^\top = I_n,$$

or that the $n \times N$ zero matrix is the unique solution of

$$\text{minimize} \;\; |V^\dagger - W|^2 \qquad \text{subject to} \quad VW^\top = 0.$$

The uniqueness is ensured by the uniqueness of best approximations from finite-dimensional subspaces of inner product spaces. To check that the zero matrix is the solution, we may verify the orthogonality characterization

$$\mathrm{Tr}(V^\dagger W^\top) = 0 \qquad \text{for any } W \text{ satisfying } VW^\top = 0.$$

We just have to remember that $V^\dagger$ is defined by

$$V^\dagger = S^{-1}V, \qquad \text{with} \quad S := VV^\top.$$

Hence, whenever $VW^\top = 0$ is satisfied, we obtain

$$\mathrm{Tr}(V^\dagger W^\top) = \mathrm{Tr}(S^{-1}VW^\top) = 0.$$

The proof now is complete. □

The following corollary applies in particular to the frames formed by the vertices of each of the five platonic solids. Indeed, they are known to constitute normalized tight frames [16]. The value of their condition numbers is provided in the table below, where $\phi = (1+\sqrt{5})/2$ is the golden ratio.

Corollary 17. For normalized tight frames, the absolute condition number reduces to the canonical condition number.

$\underline{v}$	octahedron	tetrahedron	hexahedron	icosahedron	dodecahedron
$\kappa_\infty(\underline{v})$	$\sqrt{3} \approx 1.7321$	$\sqrt{3} \approx 1.7321$	$\sqrt{3} \approx 1.7321$	$\phi \approx 1.6180$	$\dfrac{3(1+\phi)}{5} \approx 1.5708$

5.2 Allometry constants of $\mathcal{E}_n$

The standard basis for $\mathcal{E}_3$, composed of the three 'essential' vertices of the octahedron, is known to be best conditioned in $\mathcal{E}_3$. Its condition number equals $\sqrt{3}$. Since the condition number for the four vertices of the tetrahedron is not smaller than $\sqrt{3}$, we already suspect that these four vertices are not best conditioned in $\mathcal{E}_3$. Our numerical computation of the 4-th allometry constant of $\mathcal{E}_3$ confirms this fact. They also reveal that, for best conditioned 4-frames, optimal dual frames do not coincide with canonical dual frames.

The computations of the N-th allometry constant of $\mathcal{E}_n$ for various N and n are summarized in the table of values presented below. This table is consistent with the familiar value of the Banach–Mazur distance from $\mathcal{E}_n$ to ℓ_∞^n, i.e.

$$\kappa_\infty^n(\mathcal{E}_n) = \sqrt{n}.$$

Let us also mention that the column $N = \infty$ was generated by the known value — see [2] and references therein — of the projection constant of the Euclidean space $\mathcal{E}_n$, namely

$$p(\mathcal{E}_n) = \frac{2}{\sqrt{\pi}} \frac{\Gamma\big((n+2)/2\big)}{\Gamma\big((n+1)/2\big)} = \begin{cases} \dfrac{2}{\pi} \dfrac{2^{2k}}{\binom{2k}{k}}, & n = 2k, \\[2ex] (2k+1)\dfrac{\binom{2k}{k}}{2^{2k}}, & n = 2k+1. \end{cases}$$

Observe the asymptotic behavior

$$p(\mathcal{E}_n) \sim \sqrt{\frac{2}{\pi}}\,\sqrt{n} \quad \text{as } n \to \infty. \tag{6}$$

$\kappa_\infty^N(\mathcal{E}_n)$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=\infty$
$n=2$	1.4142	1.3333	1.3066	1.2944	1.2879	1.2840	1.27324
$n=3$		1.7321	1.6667	1.6300	1.5811	1.5701	1.5
$n=4$			2.0000	1.9437	1.8856	1.8705	1.69765
$n=5$				2.2361	2.1858	2.1344	1.875
$n=6$					2.4495	2.4037	2.03718

Besides the expected decrease of $(\kappa_\infty^N(\mathcal{E}_n))$ with N, we also observe an increase with n. This is easily explained. Indeed, let us consider a best conditioned N-frame $\underline{v}$ for $\mathcal{E}_{n+1}$, together with an optimal dual frame $\underline{u}$. Then, if P denotes the orthogonal projection from $\mathcal{E}_{n+1}$ onto $\mathcal{E}_n$, we notice that the frame $P\underline{u}$ is dual to the N-frame $P\underline{v}$ for $\mathcal{E}_n$, and we derive $\kappa_\infty(P\underline{v}|P\underline{v}) \le \kappa_\infty(\underline{v}|\underline{u})$, implying $\kappa_\infty^N(\mathcal{E}_n) \le \kappa_\infty^N(\mathcal{E}_{n+1})$.

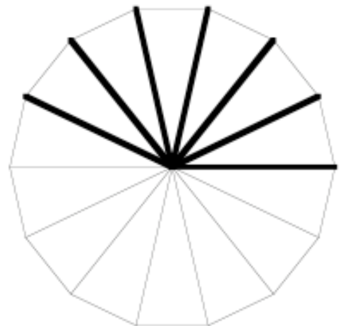


Figure 1: The best conditioned 7-frame in $\mathcal{E}_2$

We also infer from our computations that the N-th allometry constant of the Euclidean plane follows the pattern

$$\kappa_\infty^N(\mathcal{E}_2) = \frac{4/\pi}{\mathrm{sinc}(\pi/2N)} = \frac{2}{N\sin(\pi/2N)}.$$

This means that the N 'essential' vertices of the regular $2N$-gon constitute a best conditioned N-frame for $\mathcal{E}_2$ — see Figure 1. A proof of this statement is yet to be found.

6 Finite sequences with the p-norm

We examine here the allometry constants of the spaces ℓ_p^n for $p \in [1,\infty]$. The case $p=1$ is of particular interest. Because the dual unit ball of ℓ_1^n possesses 2^n vertices, Theorem 9 implies that

$$\kappa_\infty^{2^{n-1}}(\ell_1^n) = p(\ell_1^n).$$

This explains why, in the next table, the sequence $(\kappa_\infty^N(\ell_1^3))_{N\geq 3}$ is not constant but eventually stationary. The value of the projection constant $p(\ell_1^n)$ is in fact known — see [2] and references therein. Namely, we have

$$p(\ell_1^{2k-1}) = p(\ell_1^{2k}) = \frac{(2k-1)\Gamma(k-1/2)}{\sqrt{\pi}\,\Gamma(k)} = 2k\,\frac{\binom{2k}{k}}{2^{2k}}.$$

We observe an asymptotic behavior analogous to the one of (6), that is

$$p(\ell_1^n) \sim \sqrt{\frac{2}{\pi}}\,\sqrt{n} \quad \text{as } n\to\infty.$$

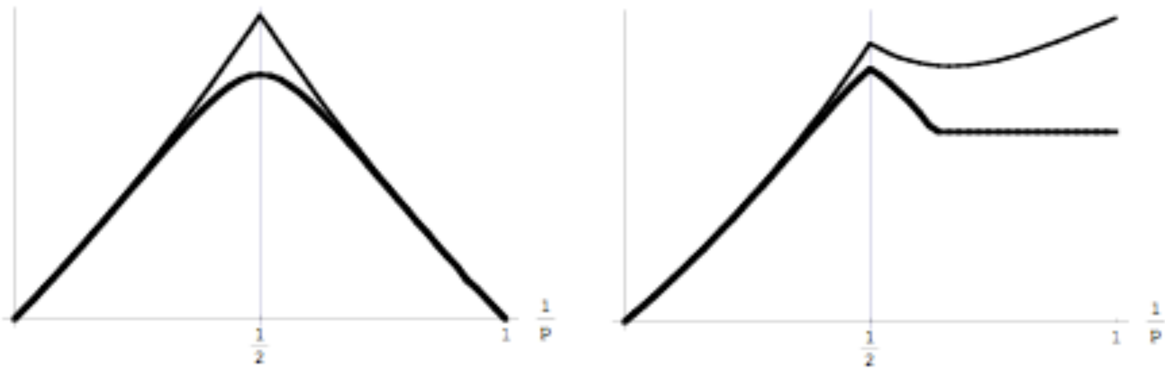


Figure 2: Left: the allometry constants $\kappa_\infty^2(\ell_p^2)$ [thin] and $\kappa_\infty^3(\ell_p^2)$ [bold]; Right: the allometry constants $\kappa_\infty^3(\ell_p^3)$ [thin] and $\kappa_\infty^4(\ell_p^3)$ [bold]

$\kappa_\infty^N(\ell_1^n)$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=\infty$
$n=2$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$n=3$		1.8000	1.5000	1.5000	1.5000	1.5000
$n=4$			2.0000	1.8889	1.7985	1.5000
$n=5$				2.3191	2.0000	1.8750
$n=6$					2.4209	1.8750

We shall now focus on the behavior of $\kappa_\infty^N(\ell_p^n)$ as a function of $p \in [1,\infty]$, or more precisely of $1/p \in [0,1]$. In Figure 2-Left, we consider the specific choices $n=2$ and $N=2,3$. We observe that the graph of $\kappa_\infty^N(\ell_p^2)$ is symmetric about $1/p = 1/2$. This readily follows from the isometry between ℓ_1^2 and ℓ_∞^2, since, for $1/p + 1/p^* = 1$, we get

$$\kappa_\infty^N(\ell_p^2) = \kappa_1^N(\ell_{p^*}^2) = \kappa_\infty^N(\ell_{p^*}^2).$$

When $n>2$, however, the spaces ℓ_1^n and ℓ_∞^n are not isometric, and the regularity of the graph of $\kappa_\infty^N(\ell_p^n)$ is lost — see Figure 2-Right for the specific choices $n=3$, $N=3,4$. We observe that $p=2$ is still a local extremum, but that it is not the only one anymore. More striking than this is the fact that $\kappa_\infty^4(\ell_p^3)$ seems to be constant when p is close to 1.

7 Algebraic polynomials with the max-norm

We focus in this section on the space $\mathcal{P}_n$ of real algebraic polynomials of degree at most n endowed with the max-norm on $[-1,1]$. For $n=1$, we note that the interpolating projection at the endpoints -1 and 1 has norm equal to 1, implying that all the allometry constants of $\mathcal{P}_1$ are equal to 1. We then concentrate on the case $n=2$.

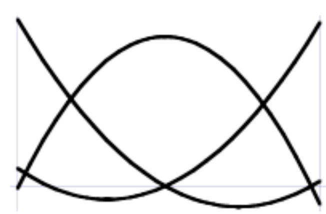
7.1 Results and their interpretation

The outcome of our computations mainly consists of the exact value of the Banach–Mazur distance from $\mathcal{P}_2$ to ℓ_∞^3, namely

$$\kappa_\infty^3(\mathcal{P}_2) \approx 1.2468.$$

This comes as a surprise, because it was anticipated that a best conditioned basis (p_1, p_2, p_3) for $\mathcal{P}_2$ could be chosen symmetric — in the sense that $p_1(-x) = p_3(x)$ and $p_2(-x) = p_2(x)$. The minimal condition number of such symmetric bases is 1.24839. We can numerically compute this value by adding extra equality constraints in the optimization problem. Besides, this value was calculated explicitly [7] under some constraints only slightly stronger than mere symmetry. As a matter of fact, it has first been obtained [4] numerically as the symmetric generalized interpolating projection constant of the space $\mathcal{P}_2$. Our computations now reveal that this value is not optimal. The basis for which the condition number 1.2468 was obtained is the nonsymmetric basis given by

$$\begin{aligned} p_1(x) &\approx \quad 0.5865x^2 - 0.5398x - 0.0127, \\ p_2(x) &\approx -1.0659x^2 - 0.0534x + 0.9993, \\ p_3(x) &\approx \quad 0.5956x^2 + 0.4864x + 0.0109. \end{aligned}$$



Even though the minimization might not be fully trusted, the calculation of the condition number of this basis can be — no minimization is involved. We indeed obtain 1.2468. This also means that no minimal generalized interpolating projection is symmetric, which is all the more surprising given that a minimal projection P can always be chosen symmetric — in the sense that $P[f(-\bullet)](x) = P[f](-x)$.

7.2 Computational outline

As was explained in Section 3, we need procedures to calculate the max-norm on $[-1,1]$ of a polynomial $p \in \mathcal{P}_n$ entered via its coefficients on a basis $\underline{w}$ for $\mathcal{P}_n$, and the norm of a linear functional λ on $\mathcal{P}_n$ entered via the values of λ on the basis $\underline{w}$. Naturally enough, the basis $\underline{w}$ is chosen to be $w_1(x) = x^n, \ldots, w_n(x) = x, w_{n+1}(x) = 1$.

Calculating the norm of $p \in \mathcal{P}_n$ is straightforward: find the critical points of p in $[-1,1]$ by determining the roots of p', and take the maximum of the absolute value of p at these points and at the endpoints. In the simple case $n = 2$, this reads

$$p(x) = ax^2 + bx + c \quad \Longrightarrow \quad \|p\|_\infty = \max\big(|a+c| + |b|, |d|\big),$$

where $d = -b^2/(4a) + c$ if the critical point $x^* := -b/(2a)$ lies in $[-1,1]$, and $d = 0$ otherwise.

Calculating the norm of a linear functional λ on $\mathcal{P}_n$ is more complicated. We use

$$\|\lambda\| = \max_{p \in \mathrm{Ex}(B_{\mathcal{P}_2})} \lambda(p),$$

combined with the characterization of the extreme points of the unit ball of $\mathcal{P}_n$ given in [10]. In the case $n = 2$, this implies

$$\mathrm{Ex}(B_{\mathcal{P}_2}) = \big\{ \pm 1 \big\} \cup \big\{ \pm P_t,\, t \in [-1,1] \big\},$$

where the quadratic polynomial P_t is defined by

$$P_t(x) := \begin{cases} \dfrac{2}{(1-t)^2}\big(x-t\big)^2 - 1 = \dfrac{1}{(1-t)^2}\big(2x^2 - 4tx + t^2 + 2t - 1\big), & t \in [-1,0], \\ \dfrac{2}{(1+t)^2}\big(x-t\big)^2 - 1 = \dfrac{1}{(1+t)^2}\big(2x^2 - 4tx + t^2 - 2t - 1\big), & t \in [0,1]. \end{cases}$$

Setting $\alpha := \lambda(w_1)$, $\beta := \lambda(w_2)$, $\gamma := \lambda(w_3)$, we subsequently obtain

$$\|\lambda\| = \max\Big(|\gamma|, \max_{t \in [-1,0]} |F_-(t)|, \max_{t \in [0,1]} |F_+(t)|\Big),$$

where the function F_- and F_+ are defined by

$$\begin{aligned} F_-(t) &:= \frac{1}{(1-t)^2}\big(2\alpha - 4t\beta + (t^2 + 2t - 1)\gamma\big), \\ F_+(t) &:= \frac{1}{(1+t)^2}\big(2\alpha - 4t\beta + (t^2 - 2t - 1)\gamma\big). \end{aligned}$$

8 Trigonometric polynomials with the max-norm

We concentrate in this section on the space $\mathcal{T}_n$ of real trigonometric polynomials of degree at most n endowed with the max-norm on the circle $\mathbb{T}$. The value of the Banach–Mazur distance from $\mathcal{T}_1$ to ℓ_∞^3, i.e. the 3-rd allometry constant of $\mathcal{T}_1$, was already found numerically and announced in [7]. Further computations, raising interesting questions, are performed here.

8.1 Results and their interpretation

To start with, we shall deal with the Banach–Mazur distance $\kappa_\infty^{2n+1}(\mathcal{T}_n)$ from the space $\mathcal{T}_n$ to the space ℓ_∞^{2n+1} — so that bases rather than spanning systems are considered. The results of Section 4 imply the chain of inequalities

$$p(\mathcal{T}_n, \mathcal{C}(\mathbb{T})) \ \le\ \kappa_\infty^{2n+1}(\mathcal{T}_n) \ \le\ p_{\rm int}(\mathcal{T}_n, \mathcal{C}(\mathbb{T}))$$

between the projection constant, the $(2n+1)$-st allometry constant, and the interpolating projection constant of the space $\mathcal{T}_n$. The Fourier projection and the interpolating projection at equidistant nodes are known — see [5] and [6] — to be minimal and minimal interpolating projections from $\mathcal{C}(\mathbb{T})$ onto $\mathcal{T}_n$, respectively. We can then infer that

$$\begin{aligned} p(\mathcal{T}_n, \mathcal{C}(\mathbb{T})) &= \quad \frac{1}{2\pi}\int_{\mathbb{T}} \left| \frac{\sin\big((2n+1)x/2\big)}{\sin\big(x/2\big)} \right| dx \quad &\sim \frac{4}{\pi^2}\ln(n), \\ p_{\rm int}(\mathcal{T}_n, \mathcal{C}(\mathbb{T})) &= \frac{1}{2n+1}\sum_{k=-n}^{n} \frac{1}{\big|\cos\big(k\pi/(2n+1)\big)\big|} &\sim \frac{2}{\pi}\ln(n). \end{aligned}$$

Note that the latter, i.e. the norm of the interpolating projection at equidistant nodes, is also the condition number of the Lagrange basis at these nodes, by virtue of Lemma 10. Our computations for $n = 1$ and $n = 2$ yield

$$\kappa_\infty^3(\mathcal{T}_1) = 1.6667, \qquad \kappa_\infty^5(\mathcal{T}_2) = 1.9889.$$

These are nothing else than the values of $p_{\rm int}(\mathcal{T}_1, \mathcal{C}(\mathbb{T}))$ and $p_{\rm int}(\mathcal{T}_2, \mathcal{C}(\mathbb{T}))$. It means that, for $n = 1$ and $n = 2$, the Lagrange bases at equidistant nodes are best conditioned. It is worth conjecturing the generalization to any n. In particular, an affirmative answer would improve on the minimality of the interpolating projection at equidistant nodes established in [6].

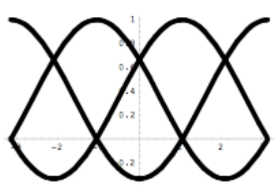
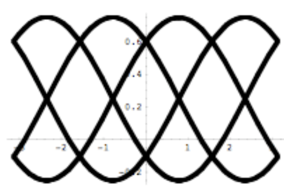
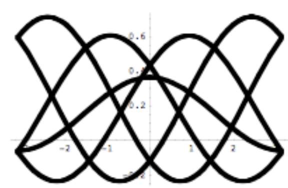
Conjecture. The Lagrange basis $\underline{\ell} = (\ell_{-n}, \ldots, \ell_0, \ldots, \ell_n)$ relative to the equidistant points $\tau_k := k\,\dfrac{2\pi}{2n+1}$, $k \in [\![-n, n]\!]$, is best conditioned for the space $\mathcal{T}_n$.

Let us mention a nice feature of the basis $\underline{\ell}$, namely that it is orthogonal with respect to the inner product on $L_2(\mathbb{T})$. This can be seen from the fact that the functions ℓ_k are translates of the function ℓ_0, and that the latter reduces to the kernel of the Fourier projection, up to a multiplicative constant, specifically

$$\ell_0(x) = \frac{1}{2n+1}\frac{\sin\big((2n+1)x/2\big)}{\sin\big(x/2\big)}.$$

One can also use this to establish that the projection constant $p(\mathcal{T}_n, \mathcal{C}(\mathbb{T}))$ is the average of the Lebesgue function whose maximum is the interpolating projection constant $p_{\mathrm{int}}(\mathcal{T}_n, \mathcal{C}(\mathbb{T}))$.

Let us now return to the general case of spanning systems. The following table displays the values of the N-th allometry constants of the space $\mathcal{T}_1$ for $N = 3$, $N = 4$, and $N = 5$, together with the best conditioned N-systems generated by our computations. We observe that the best conditioned 3-systems is composed of translates of the kernel of the Fourier projection, as already outlined. This also seems to be the case for $N = 4$, but surprisingly not for $N = 5$.

	$N = 3$	$N = 4$	$N = 5$
$\kappa_\infty^N(\mathcal{T}_1)$	1.6667	1.5000	1.4716
best conditioned N-system			

8.2 Computational outline

We need procedures to calculate the max-norm on $\mathbb{T}$ of a trigonometric polynomial $q \in \mathcal{T}_n$ entered via its coefficients on a basis $\underline{w}$ of $\mathcal{T}_n$, and the norm of a linear functional λ on $\mathcal{T}_n$ entered via the values of λ on the basis $\underline{w}$. This basis is chosen to be

$$w_1(x) = \cos(nx),\ w_2(x) = \sin(nx),\ \ldots,\ w_{2n-1}(x) = \cos(x),\ w_{2n}(x) = \sin(x),\ w_{2n+1}(x) = 1.$$

As it happens, the case $n = 1$ is straightforward, since

$$q(x) = a\cos(x) + b\sin(x) + c \quad \Longrightarrow \quad \|q\|_\infty = \sqrt{a^2+b^2} + |c|,$$

and $\|\lambda\| = \max\left\{a\lambda(\cos) + b\lambda(\sin) + c\lambda(1),\ \sqrt{a^2+b^2} + |c| = 1\right\}$ then yields

$$\|\lambda\| = \max\left(|\lambda(1)|, \sqrt{\lambda(\cos)^2 + \lambda(\sin)^2}\right).$$

In contrast, the case $n = 2$ requires more attention. We separate the determinations of the norm and of the dual norm in two subsections.

8.2.1 Norm on the space $\mathcal{T}_2$

We wish to determine the max-norm of a trigonometric polynomial $q \in \mathcal{T}_2$ given by

$$q(x) = a\cos(2x) + b\sin(2x) + c\cos(x) + d\sin(x) + e.$$

We have to find the critical points by solving the equation

$$\begin{aligned} 0 = q'(x) &= -2a\sin(2x) + 2b\cos(2x) - c\sin(x) + d\cos(x) \\ &= -4a\cos(x)\sin(x) + 2b[2\cos^2(x) - 1] - c\sin(x) + d\cos(x). \end{aligned}$$

1/ If $x \in [0, \pi]$, we set $u := \cos(x) \in [-1, 1]$, so that $\sin(x) = \sqrt{1-u^2}$. The equation reads

$$0 = -4au\sqrt{1-u^2} + 2b(2u^2 - 1) - c\sqrt{1-u^2} + du.$$

After rearranging the terms, we obtain the equation

$$(4au + c)\sqrt{1-u^2} = 4bu^2 + du - 2b.$$

Squaring the latter and collecting the powers of u, we arrive at the quadric equation

$$C_1u^4 + C_2u^3 + C_3u^2 + C_4u + C_5 = 0,$$

where the coefficients are given by

$$\begin{aligned} C_1 &:= \quad 16(a^2+b^2), \\ C_2 &:= \quad 8(bd + ac), \\ C_3 &:= -16(a^2+b^2) + c^2 + d^2, \\ C_4 &:= -8ac - 4bd, \\ C_5 &:= \quad 4b^2 - c^2. \end{aligned}$$

The solutions — rejecting those outside $[-1,1]$ — are obtained using MATLAB's function `roots`.

2/ If $x \in [-\pi, 0]$, we set $u := \cos(x) \in [-1,1]$, so that $\sin(x) = -\sqrt{1-u^2}$. The critical points satisfy the previous equation modulo the changes $a \leftrightarrow -a$ and $c \leftrightarrow -c$, leading to the very same quadric equation.

8.2.2 Dual norm on the space $\mathcal{T}_2$

Our argument relies on the description of the extreme points of the unit ball of the space $\mathcal{T}_n$. The proof of Theorem 18 — omitted — follows the ideas of [10] closely, and is in fact easier, because we need not worry about endpoints. Proposition 19 follows without much difficulty.

Theorem 18. A trigonometric polynomial $q \in \mathcal{T}_n$ with $\|q\|_\infty = 1$ is an extreme point of the unit ball of $\mathcal{T}_n$ if and only if, counting multiplicity, the numbers of zeros of $q-1$ and of $q+1$ sum at least to $2n+2$.

Proposition 19. The set of extreme points of the unit ball of $\mathcal{T}_2$ is explicitly given by

$$\mathrm{Ex}(B_{\mathcal{T}_2}) = \Big\{ \pm 1 \Big\} \cup \Big\{ \pm Q_{D,t},\ D \in [-1,1],\ t \in \mathbb{T} \Big\},$$

where the trigonometric polynomial $Q_{D,t}$ is defined by

$$\begin{aligned} Q_{D,t}(x) &:= \frac{2}{(1+|D|)^2}\big(\cos(x-t) - D\big)^2 - 1 \\ &= \frac{\cos(2t)\cos(2x) + \sin(2t)\sin(2x) - 4D\cos(t)\cos(x) - 4D\sin(t)\sin(x) + D^2 - 2|D|}{(1+|D|)^2}. \end{aligned}$$

For a linear functional λ on $\mathcal{T}_2$ entered via the values $\alpha = \lambda(w_1)$, $\beta = \lambda(w_2)$, $\gamma = \lambda(w_3)$, $\delta = \lambda(w_4)$, and $\varepsilon = \lambda(w_5)$, we may now write

$$\|\lambda\| = \max_{q \in \mathrm{Ex}(B_{\mathcal{T}_2})} \lambda(q) = \max\Big(|\varepsilon|, \max_{D\in[-1,1],\ t\in\mathbb{T}} |F(D,t)| \Big),$$

where the function F is defined by

$$F(D,t) := \frac{\alpha\cos(2t) + \beta\sin(2t) - 4D\gamma\cos(t) - 4D\delta\sin(t) + \varepsilon(D^2 - 2|D|)}{(1+|D|)^2}.$$

Observe that $F(-D,t+\pi)=F(D,t)$, so that the norm of λ reduces to

$$\|\lambda\| = \max\big(\,|\varepsilon|, \max_{D\in[0,1],\, t\in\mathbb{T}} |F(D,t)|\,\big).$$

Note that $\max_{t\in\mathbb{T}}|F(0,t)|$ and $\max_{t\in\mathbb{T}}|F(1,t)|$ are computed as the max-norms of some trigonometric polynomials in $\mathcal{T}_2$. We also have to take into account the values of F at the critical points in $(0,1)\times\mathbb{T}$. These are determined by the system of equations

$$\begin{aligned}
&\frac{\partial F}{\partial t}(D,t)=0, \quad \text{i.e.} \quad 2D(\gamma\sin(t)-\delta\cos(t)) = \alpha\sin(2t)-\beta\cos(2t),\\
&\frac{\partial F}{\partial D}(D,t)=0, \quad \text{i.e.} \quad 2D(\gamma\cos(t)+\delta\sin(t)+\varepsilon) = \alpha\cos(2t)+\beta\sin(2t)+2\gamma\cos(t)+2\delta\sin(t)+\varepsilon.
\end{aligned}$$

1/ If $t\in[0,\pi]$, we set $u:=\cos(t)$, so that $\sin(t)=\sqrt{1-u^2}=:\surd$. Eliminating D, we get

$$\big(\alpha 2u\surd-\beta(2u^2-1)\big)\big(\gamma u+\delta\surd+\varepsilon\big) = \big(\alpha(2u^2-1)+\beta 2u\surd+2\gamma u+2\delta\surd+\varepsilon\big)\big(\gamma\surd-\delta u\big).$$

After rearranging the terms, we obtain

$$\begin{aligned}
&\big(2(\alpha\gamma-\beta\delta)u^2+2\alpha\varepsilon u+\beta\delta\big)\surd-2(\beta\gamma+\alpha\delta)u^3-2\beta\varepsilon u^2+(\beta\gamma+2\alpha\delta)u+\beta\varepsilon\\
=\;&\big(2(\alpha\gamma-\beta\delta)u^2+2(\gamma^2-\delta^2)u-\alpha\gamma+\gamma\varepsilon\big)\surd-2(\alpha\delta+\beta\gamma)u^3-4\alpha\delta u^2+(\alpha\delta+2\beta\gamma-\delta\varepsilon)u+2\delta\gamma,
\end{aligned}$$

or, in a simpler form,

$$\big(2(\alpha\varepsilon+\delta^2-\gamma^2)u+(\beta\delta+\alpha\gamma-\gamma\varepsilon)\big)\surd = 2(\beta\varepsilon-2\alpha\delta)u^2+(\beta\gamma-\alpha\delta-\delta\varepsilon)u+(2\delta\gamma-\beta\varepsilon).$$

Squaring the latter and collecting the powers of u, we arrive at the quadric equation

$$C_1u^4+C_2u^3+C_3u^2+C_4u+C_5=0,$$

where the coefficients are given by

$$\begin{aligned}
C_1 &:= -4\big[(\alpha\varepsilon+\delta^2-\gamma^2)^2+(\beta\varepsilon-2\alpha\delta)^2\big],\\
C_2 &:= -4\big[(\alpha\varepsilon+\delta^2-\gamma^2)(\beta\delta+\alpha\gamma-\gamma\varepsilon)+(\beta\varepsilon-2\alpha\delta)(\beta\gamma-\alpha\delta-\delta\varepsilon)\big],\\
C_3 &:= \;\;4\big[(\alpha\varepsilon+\delta^2-\gamma^2)^2-(\beta\varepsilon-2\alpha\delta)(2\delta\gamma-\beta\varepsilon)\big]-\big[(\beta\delta+\alpha\gamma-\gamma\varepsilon)^2+(\beta\gamma-\alpha\delta-\delta\varepsilon)^2\big],\\
C_4 &:= \;\;4\big[(\alpha\varepsilon+\delta^2-\gamma^2)(\beta\delta+\alpha\gamma-\gamma\varepsilon)\big]-2\big[(\beta\gamma-\alpha\delta-\delta\varepsilon)(2\delta\gamma-\beta\varepsilon)\big],\\
C_5 &:= \;\;\big[(\beta\delta+\alpha\gamma-\gamma\varepsilon)^2-(2\delta\gamma-\beta\varepsilon)^2\big].
\end{aligned}$$

2/ If $t\in[-\pi,0]$, we set $u:=\cos(t)$, so that $\sin(t)=-\sqrt{1-u^2}$. The critical point (D,t) satisfy the previous system of equations modulo the changes $\beta\leftrightarrow-\beta$, $\delta\leftrightarrow-\delta$, leading to the very same quadric equation.

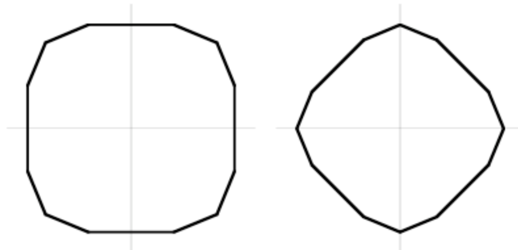


Figure 3: Unit ball and dual unit ball of the Pan–Shekhtman space

9 Polygonal spaces

By a **polygonal space**, we understand the two-dimensional space $\mathbb{R}^2$ endowed with a norm that makes its unit ball — or its dual unit ball — a convex symmetric polygon. For example, the space $\mathcal{V}$ whose unit ball and dual unit ball are represented in Figure 3 is normed by

$$\left\| \, [a,b]^\top \right\| = \max\big(|a|, |b|, \alpha|a| + \beta|b|, \beta|a| + \alpha|b|\big), \qquad \alpha := \frac{2+\sqrt{2}}{4}, \quad \beta := \frac{\sqrt{2}}{4}.$$

It was introduced by Pan and Shekhtman [13] as an example of a space whose projection constant and relative interpolating projection constant are equal — as a matter of fact, projection constant and *absolute* interpolating projection constant are equal, see [8]. The inequalities $p(\mathcal{V}) \le \kappa_\infty^N(\mathcal{V}) \le p_{\mathrm{int}}(\mathcal{V})$ for $N \ge 2$ then imply that $\kappa_\infty^N(\mathcal{V})$ is independent of N. We can use our computations to verify this numerically: first, check that $\kappa_\infty^N(\mathcal{V}) = (1+\sqrt{2})/2$ for $N < 6$, and then invoke Theorem 9 for $N \ge 6$.

9.1 Results and their interpretations

We only mention in this subsection the special polygonal space $\mathcal{G}_m$ whose unit ball is the regular $2m$-gon. According to the following table of values — see also Figure 4 — we notice the yet unjustified pattern

$$\begin{array}{llll} \kappa_\infty^2(\mathcal{G}_{4k-2}) < \sqrt{2}, & \kappa_\infty^2(\mathcal{G}_{4k}) = \sqrt{2}, & \kappa_\infty^2(\mathcal{G}_{4k\pm1}) > \sqrt{2}, & k \in \mathbb{N}. \\ & \kappa_\infty^3(\mathcal{G}_{3k}) = 4/3, & \kappa_\infty^3(\mathcal{G}_{3k\pm1}) < 4/3, & k \in \mathbb{N}. \end{array}$$

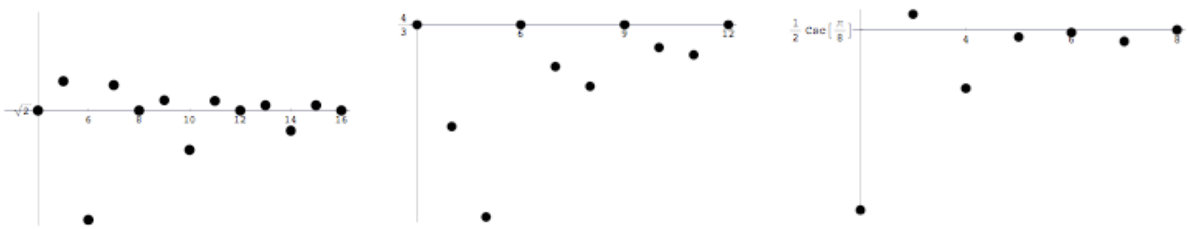


Figure 4: 2-nd, 3-rd, and 4-th allometry constants of the spaces $\mathcal{G}_m$ as functions of m

We also remark that the limiting values $\sqrt{2}$ and $4/3$ are the 2-nd and 3-rd allometry constants $\kappa_\infty^2(\mathcal{E}_2)$ and $\kappa_\infty^3(\mathcal{E}_2)$ of the Euclidean plane $\mathcal{E}_2$, which was intuitively expected.

$\kappa_\infty^N(\mathcal{G}_m)$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$\kappa_\infty^N(\mathcal{E}_2)$
$N=2$	1.0000	1.5000	1.4142	1.4271	1.3660	1.4254	1.4142	1.4142
$N=3$	1.0000	1.3333	1.3204	1.3090	1.3333	1.3280	1.3255	1.3333
$N=4$	1.0000	1.3333	1.2071	1.2944	1.3022	1.2871	1.3066	1.3066

The results are the same for the polygonal space whose dual unit ball is the regular $2m$-gon. Indeed, the space $\mathcal{G}_m$ is 'self-dual', which means that the dual space $\mathcal{G}_m^*$ is isometric to the space $\mathcal{G}_m$ itself — see next subsection. This also implies that

$$\kappa_1^N(\mathcal{G}_m) = \kappa_\infty^N(\mathcal{G}_m).$$

9.2 Computational outline

Let the polygonal space $\mathcal{V}$ be determined by the vertices of its unit ball, i.e. by the points

$$\nu_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \ldots, \quad \nu_{2m} = \begin{pmatrix} x_{2m} \\ y_{2m} \end{pmatrix}.$$

The norm of a dual functional λ on $\mathcal{V}$ is easy to determine, since

$$\|\lambda\| = \max_{v \in \mathrm{Ex}(B_\mathcal{V})} \lambda(v) = \max_{k \in [\![1,2m]\!]} \lambda(\nu_k) = \max_{k \in [\![1,2m]\!]} \big(\alpha x_k + \beta y_k\big), \qquad \alpha := \lambda(\,[1,0]^\top),\ \beta := \lambda(\,[0,1]^\top).$$

As for the norm of a vector $v = [a,b]^\top$ in the space $\mathcal{V}$, it is given by

$$\|v\| = \max_{k \in [\![1,2m]\!]} \langle v, u_k \rangle = \max_{k \in [\![1,2m]\!]} \big(a\widetilde{x}_k + b\widetilde{y}_k \big),$$

where the line of equation $\langle v, u_k \rangle = 1$ is the line passing through the vertices ν_k and ν_{k+1}, thus

$$u_k = \begin{pmatrix} \widetilde{x}_k \\ \widetilde{y}_k \end{pmatrix} = \frac{1}{x_k y_{k+1} - x_{k+1} y_k} \begin{pmatrix} y_{k+1} - y_k \\ x_k - x_{k+1} \end{pmatrix}.$$

In view of the analogy between the norm and the dual norm modulo the changes $\nu_k \leftrightarrow u_k$, the computations for polygonal spaces determined by the vertices of their dual unit balls are not different. Note also that in a regular $2m$-gon, the points u_k are simply the midpoints of the vertices ν_k and ν_{k+1}, hence they also represent the vertices of a regular $2m$-gon. This accounts for the isometry, previously mentioned, between the space $\mathcal{G}_m$ and its dual.

10 Closing remarks

As a conclusion to the paper, we reformulate two classical questions about Banach–Mazur distances and projection constants in terms of allometry constants.

10.1 Extremality of the hexagonal space

It is known [1] that the space $\mathcal{G}_3$, whose unit ball is a regular hexagon, maximizes the Banach–Mazur distance $\kappa_\infty^2(\mathcal{V})$ from $\mathcal{V}$ to ℓ_∞^2 over all two-dimensional normed spaces $\mathcal{V}$. It was also asserted [9] that it maximizes the projection constant $p(\mathcal{V})$, but Chalmers [3] claimed that this original proof is incorrect. It is nonetheless natural to wonder if the hexagonal space maximizes all the intermediate quantities, i.e. all $\kappa_\infty^N(\mathcal{V})$ for $N \ge 2$. It need only be established for $N = 3$. Indeed, for $N \ge 3$, it would imply

$$\kappa_\infty^N(\mathcal{V}) \le \kappa_\infty^3(\mathcal{V}) \le \kappa_\infty^3(\mathcal{G}_3) = p(\mathcal{G}_3) \le \kappa_\infty^N(\mathcal{G}_3).$$

Taking the limit $N \to \infty$ would in particular yield the inequality $p(\mathcal{V}) \le p(\mathcal{G}_3)$.

10.2 Bounding allometry constants in terms of projection constants

As a counterpart to the inequality $p(\mathcal{V}) \leq \kappa_\infty^n(\mathcal{V})$ for n-dimensional spaces $\mathcal{V}$, the possibility of an inequality $\kappa_\infty^n(\mathcal{V}) \leq C \cdot p(\mathcal{V})$ for some absolute constant C was explored for some time and then refuted in [15]. But fixing the constant C — taking $C = 2$, say — there exists $N = N(\mathcal{V})$ for which $\kappa_\infty^N(\mathcal{V}) \leq C \cdot p(\mathcal{V})$. It would be interesting to know how small N can be taken, i.e. to estimate the integer $N(n)$ for which the inequality $\kappa_\infty^{N(n)}(\mathcal{V}) \leq C \cdot p(\mathcal{V})$ holds independently of the n-dimensional space $\mathcal{V}$.

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