Name ______________________________ **I**

Mid-Term—Math 308-509

Instructions: *Show all work on the paper provided. Only scientific calculators are allowed.*

PART I

1. **(10 pts.)** Find the solution to $y' = (9 - y^2)\sin t$, $y(\pi) = 0$.

2. **(5 pts.)** If L is the operator $L = D^2 - xD + 3$, find $L[e^{2x}]$.

3. **(15 pts.)** Compute the Wronskian of the set $\{xe^{-2x}, xe^{-3x}\}$. Is the set linearly dependent or linearly independent? Can $\{xe^{-2x}, xe^{-3x}\}$ be a fundamental set for $L[y] = y'' + p(x)y' + q(x)y = 0$, with p and q being continuous for all x? Explain.

4. **(10 pts.)** Verify that the equation $y^2 dx + (2xy - \sin(y))dy = 0$ is exact, and solve the IVP comprising this equation together with $y(1) = \pi$.

5. **(10 pts.)** The Gompertz equation, which is given below, is used to model population growth.

$$\frac{dp}{dt} = -rp\ln(p/K), \quad r > 0, K > 0 \text{ are constants, and } p \geq 0$$

Find the equilibrium points. For what points (t_0, p_0) are we assured that the associated IVP with initial conditions $p(t_0) = p_0$ can always be uniquely solved? At $p(3) = K/2$, is the population increasing or decreasing? Can the population go from $p(3) = K/2$ to $p = 2K$? Explain.

6. **(10 pts.)** Use reduction of order to find a second linearly independent solution to $x^2y'' - 5xy' + 9y = 0$, given that $y_1 = x^3$ is a solution.

PART II

1. **(20 pts.)** The heat in a building is turned on at 6 am and set to 68^0. At that time the building's temperature is 55^0; the outside temperature is 45^0, and remains so the entire day. If the time constant for the building is 2 hours and if the building's time constant with heating or cooling is 1/3 hours, then what is the temperature in the building at 10 am?

2. **(20 pts.)** A nitric acid solution flows at a constant rate of 6 L/min into a large tank that initially held 200 L of a 0.5% nitric acid solution. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 8 L/min. If the solution entering the tank is 20% nitric acid, determine the volume of nitric acid at any time $t > 0$.