Test I

1. Solve the initial value problems below.

   (a) **(10 pts.)** $y' = (4 + y^2)\sin t,\ y(\pi) = 0\,.$

   (b) **(10 pts.)** $(\cos x)y' - (\sin x)y = x\cos x,\ y(\pi) = 0.$

2. **(15 pts.)** Let $L[y] = y'' - y' - 6y$. Find $L[e^{rx}]$, and determine a fundamental set of solutions to $L[y] = 0$. Solve the initial value problem $L[y] = 0$, $y(0) = 1$, and $y'(0) = -1$.

3. **(15 pts.)** A population model is governed by the logistic equation,

$$\frac{dP}{dt} = 3P - P^2 \quad \text{where } P \geq 0.$$

   Use `dfield5` to find the equilibrium points (constant solutions), and to determine whether each is stable or unstable. Can the population increase from 1.5 to 2.2, or from 2.9 to 3.1? As $t \to +\infty$, does $P(t)$ approach a limit?

4. Let $L[y] = x^2y'' - xy' + y$. You are given that $y_1(x)$ satisfies $L[y_1] = 0$.

   (a) **(10 pts.)** For any function $u(x)$, find $L[u\,y_1]$.

   (b) **(10 pts.)** Use the result above and reduction of order to find a second solution to $L[y] = 0$, given that $y_1(x) = x$.

5. **(7 pts.)** You are given that $y_1 = \cos(x)$ and $y_2 = \sin(\pi/2 - x)$ satisfy $y'' + y = 0$. Calculate the Wronskian $W[y_1, y_2]$ and evaluate it at $x = 0$. From your result, show that the trigonometric identity $\cos(x) = \sin(\pi/2 - x)$ holds.

6. **(23 pts.)** A 20% alcohol solution flows into a tank at a constant rate of 5 L/min. The solution inside the tank is kept well stirred and flows out of the tank at the same rate. Given that the tank initially held 200 L of a 1% alcohol solution, determine the concentration of alcohol in the tank at any time $t > 0$. Use MATLAB to make a plot of the concentratration. Either graphically or analytically, determine the time when the concentration reaches 7%. Indicate this point on your plot.