Test I

Instructions: *Show all work in your bluebook. Calculators that do linear algebra or calculus are* not *allowed.*

1. For triangle $\triangle PQR$ with vertices $P(1,0,1)$, $Q(1,2,2)$, $R(0,1,1)$, find the following things:
   (a) **(5 pts.)** The parametric equation for the plane containing $\triangle PQR$.
   (b) **(5 pts.)** $\cos(\angle RPQ)$
   (c) **(5 pts.)** The area of $\triangle PQR$.
2. **(15 pts.)** Let $\mathbf{x} = (1, -4, 2, -1)$ and $\mathbf{v} = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$. Find the component of $\mathbf{x}$ in the direction of $\mathbf{v}$ (i.e., the projection $\mathbf{p}$ of $\mathbf{x}$ on $\mathbf{v}$), and the component $\mathbf{q}$ of $\mathbf{x}$ perpendicular to $\mathbf{v}$.
3. **(15 pts.)** Determine whether the vectors $\mathbf{a}_1 = (-1\ -2\ 1\ 2)^T$, $\mathbf{a}_2 = (2\ 1\ 1\ 0)^T$, $\mathbf{a}_3 = (3\ 3\ 0\ -2)^T$ are linearly independent (LI) or linearly dependent (LD). If they are LD, find at least one nonzero linear combination of them that is $\mathbf{0}$.
4. A linear system $A\mathbf{x} = \mathbf{b}$ has the augmented matrix $[A|\mathbf{b}]$ given below.
$$[A|b] = \left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & 1 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & -2 & -1 \end{array}\right)$$
   (a) **(10 pts.)** Put $[A|\mathbf{b}]$ in reduced row echelon form. Determine $\operatorname{rank}(A)$ and $\operatorname{rank}([A|\mathbf{b}])$. Is the system consistent or inconsistent? If it is consistent, find the parametric form of the solution.
   (b) **(5 pts.)** The corresponding homogeneous system is $A\mathbf{x} = \mathbf{0}$. Are there any nonzero solutions? Are the columns of A linearly independent or dependent?
5. **(15 pts.)** Use row reduction either to find C^{-1} or to show that it does not exist, given that the matrix C is
$$C = \begin{pmatrix} 2 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}.$$

Please turn over.

6. **(15 pts.)** Given the matrix B below, use any method to evaluate $\det(B)$. From your answer, determine whether B has an inverse. If it does, use Cramer's rule to find the (3,1)-entry the inverse.

$$B = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 1 & -3 & 0 & 2 \\ 1 & -2 & 1 & 0 \\ -2 & 0 & -2 & 3 \end{pmatrix}.$$

7. **(10 pts.)** Do *one* of the following problems.

   (a) Starting with the properties of the dot product in $\mathbb{R}^n$ – positivity, symmetry, additivity, and homogeneity –, prove Schwarz's inequality.

   (b) Show that if A and B are $m \times p$ and $p \times n$ matrices, then $(AB)^T = B^T A^T$.

   (c) Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $f(\mathbf{x})$ can be written as $f(\mathbf{x}) = A\mathbf{x}$, where $A = [f(\mathbf{e}_1) \cdots f(\mathbf{e}_n)]$.

   (d) Let $\vec{a}$ be a fixed vector in $\mathbb{R}^3$, and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(\vec{x}) = \vec{a} \times \vec{x}$. Show that f is linear and find the matrix A from part 7c that represents f.

Bonus (5 pts.) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. Observe that $A^2 = I$. Use this to calculate e^{tA}.