Final Exam – Part 1

Instructions: *This take-home part is due on Tuesday, 12/12/06. You may not get help on the test from anyone except your instructor.*

1. Let $S^h(3,2)$ be the space of C^2 piecewise cubic splines on $[0,1]$, with $x_j = jh$, $h = 1/n$, $j = 0 \ldots n$.

   (a) **(5 pts.)** For any j, $1 \le j \le n-1$ and any $f \in S^h(3,2)$, let q and Q be the cubic polynomials that agree with f to the left and the right of x_j, respectively. Specifically, these are defined by $q(x) := f(x)$ when $x_{j-1} \le x \le x_j$ and $Q(x) := f(x)$ when $x_j \le x \le x_{j+1}$. Show that as cubic polynomials, $q(x) - Q(x) = A_j(x - x_j)^3$, where A_j is a constant independent of x.

   (b) **(5 pts.)** Let $2 < j < n-2$. Find $f(x) \in S^h(3,2)$ such that $f(x_j) = 1$ and $f(x) = 0$ if $x \le x_{j-2}$ or if $x \ge x_{j+2}$. Your answer has a very simple form if you use the function $(\cdot)_+$ defined by

   $$(x)_+ := \begin{cases} x & \text{if } x \ge 0, \\ 0 & \text{if } x < 0. \end{cases}$$

   (c) **(5 pts.)** Take $n = 6$ and $y_j = \sin(2\pi x_j)$. Use Matlab or similar software to interpolate the points (x_j, y_j) using a linear spline and a cubic spline. Plot these two curves along with $y = \sin(2\pi x)$. Constructing the cubic spline requires specifying $n+3 = 9$ parameters. However, you are only supplying $n+1 = 7$ pieces of data. Explain how the software you are using handles this problem.

2. **(10 pts.)** Prove Theorem 3.9 (p. 124) in the text.

3. **(10 pts.)** Find a Green's function for the boundary value problem $Lu = -u" + u$, $u \in L^2([0,\infty))$, $u(0) = 0$. (Hint: modify the argument used in the example at the end of §4.2.)

4. The DFT is used to compute approximations to Fourier series coefficients and to Fourier transforms. The idea is to do the following. A signal is really of finite duration, $t = 0$ to $t = T$. Its Fourier transform on this interval is

   $$\hat{f}(\omega) = \int_0^T f(t)e^{-i\omega t}dt.$$

(a) **(5 pts.)** If $f(t)$ is sampled at n equally spaced points, $t_j = (T/n)j$, $j = 0, \ldots n-1$, and if $y_j = f(t_j)$, then show that $\hat{f}$ is related to the the DFT of y_j by the formula,

$$\hat{f}(\omega_k) \approx \frac{T}{n} \hat{y}_k, \tag{1}$$

where $\omega_k = \frac{2\pi}{T} k$, $k = 0, \ldots n-1$.

(b) **(5 pts.)** Calculate the Fourier transform $\hat{f}$ of $f(t) = N_2(t)$, where N_2 is the linear B-spline (tent function) with support on $[0, 2]$. Note that for any $T \geq 2$, we have

$$\hat{f}(\omega) = \int_0^2 N_2(t) e^{-i\omega t} dt = \int_0^T N_2(t) e^{-i\omega t} dt.$$

(c) **(5 pts.)** Use Matlab or some other software to numerically calculate the FFT approximation to $\hat{f}$ for $n = 1024$ and $T = 16$. Plot $|\hat{f}(\omega)|^2$ and compare it to $|\frac{T}{n} \hat{y}_k|^2$. Explain why the curves look so different in the second half of the interval.