Adjoints and Self-Adjoint Operators
Finite Dimensional Case

Francis J. Narcowich

September 2014

1 Definition of the Adjoint

Let V and W be real or complex finite dimensional vector spaces with inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$, respectively. Let $L : V \to W$ be linear. If there is a transformation $L^* : W \to V$ for which

$$\langle Lv, w \rangle_W = \langle v, L^* w \rangle_V \tag{1}$$

holds for every pair of vectors $v \in V$ and w in W, then L^* is said to be the *adjoint* of L. Some of the properties of L^* are listed below.

Proposition 1.1. *Let $L : V \to W$ be linear. Then L^* exists, is unique, and is linear.*

Proof. Introduce an orthonormal basis B for V and C for W. Then, relative to these bases, the matrix for L is

$$A_L = \big[[Lv_1]_C \; [Lv_2]_C \; \cdots \; [Lv_n]_C \big]. \tag{2}$$

Also, relative to B and C, it is easy to show that the inner products become

$$\langle u, v \rangle_V = [v]_B^* [u]_B \text{ and } \langle w, r \rangle_W = [r]_C^* [w]_C$$

From this and standard matrix algebra, it follows that

$$\langle Lv, w \rangle_W = [w]_C^* A_L [v]_B = (A_L^* [w]_C)^* [v]_B.$$

Of course, $A_L^* \in \mathbb{C}^{n \times m}$ exists, is unique, and takes $\mathbb{C}^m \to \mathbb{C}^n$. Now, let $y = [w]_C$ and set $x = A_L^* y$. Define $v = \sum_{j=1}^n x_j u_j$, so that $x = [v]_B$

and, consequently, $[v]_B = A_L^*[w]_C$. This uniquely defines a (linear) map $L^* : W \to V$. Moreover,

$$\langle L^*w, v\rangle_V = [v]_B^* A_L^*[w]_C = (A_L[v]_B)^*[w]_C = \langle w, Lv\rangle_W.$$

Taking conjugates above yields $\langle Lv, w\rangle_W = \langle v, L^*w\rangle_V$. Thus, L^* exists and, obviously, is unique. ∎

It is worthwhile to formally state a result that we actually got in the course of establishing the Proposition 1.1 results above.

Corollary 1.2. *Let V and B be as described above. If $L : V \to V$ is a linear transformation whose matrix relative to B is A_L, then the matrix of L^* is $A_{L^*} = A_L^*$.*

We say that $L : V \to V$ is *self adjoint* if and only if $L^* = L$. Self-adjoint transformations are extremely important; we will discuss some of their properties later. Before we do that, however, we should look at a few examples of adjoints for linear transformations.

Example 1.3. Consider the usual inner product on $V = \mathbb{C}^n$; this is given by $\langle x, y\rangle_{\mathbb{C}^N} = y^*x$. As noted above, for an $n \times n$ matrix A, $\langle Ax, y\rangle_{\mathbb{C}^N} = y^*Ax = (A^*y)^*x$. Thus A^* is the conjugate transpose of A, a fact we tacitly used above.

Example 1.4. Let $V = P_n$, where we allow the coefficients of the polynomials to be complex valued. For an inner product, take

$$\langle p, q\rangle = \int_{-1}^{1} p(x)\overline{q(x)}dx\,, \tag{5}$$

and for L take

$$L(p) = [(1 - x^2)p']'\,. \tag{6}$$

Doing an integration by parts yields $\langle p, Lq\rangle = \langle Lp, q\rangle$. Thus, $L = L^*$ and L is self adjoint.

Example 1.5. Let V be the set of all complex valued polynomials that are of degree n or less. Let $L(p) = xp'$ and use $\langle p, q\rangle = \int_0^1 p(x)\overline{q(x)}dx$ as the inner product. Again, an integration by parts shows that

$$\int_0^1 xp'(x)\overline{q(x)}dx = p(1)\overline{q(1)} + \int_0^1 p(x)\overline{(-xq(x))'}dx.$$

It would seem that $L^*q = -(xq(x))'$. This isn't quite correct, however. We need to take care of the term $p(1)\overline{q(1)}$. To do this, we begin by finding a polynomial $\delta(x)$[1] such that $\int_0^1 p(x)\delta(x)dx = p(1)$, for all $p \in V$. This can be always be done. For example, if $n = 2$, one can show that $\delta(x) = 3 - 24x + 30x^2$. In any case, we see that $p(1)\overline{q(1)} = \int_0^1 p(x)\overline{q(1)\delta(x)}dx$. Hence,

$$\int_0^1 xp'(x)\overline{q(x)}dx = \int_0^1 p(x)\overline{(-xq(x))' + q(1)\gamma(x)}dx.$$

It follows that $L^*q(x) = -(xq(x))' + q(1)\delta(x)$.

2 Spectral Theory for Self-Adjoint Operators

Having done a few examples, let us return to our discussion of self-adjoint transformations. We begin with the general case where the vector space V is not assumed to be finite dimensional. We have the following important result.

Proposition 2.1. *Let V be a complex vector space with an inner product. If $L : V \to V$ is a self-adjoint linear transformation, then the eigenvalues of L are real numbers, and eigenvectors of L corresponding to distinct eigenvalues are orthogonal.*

Proof. Suppose that λ is an eigenvalue of L and that x is a corresponding eigenvector. We therefore have $Lx = \lambda x$, and so $\langle Lx, x\rangle = \langle \lambda x, x\rangle = \lambda\langle x, x\rangle$. Similarly, we see that $\langle x, Lx\rangle = \langle x, \lambda x\rangle = \bar{\lambda}\langle x, x\rangle$. Now, because $L = L^*$, we have that $\langle Lx, x\rangle = \langle x, Lx\rangle$, which together with the previous two equations gives us $\lambda\langle x, x\rangle = \bar{\lambda}\langle x, x\rangle$. Finally, since $x \neq 0$, we may divide by $\langle x, x\rangle$; the result is $\lambda = \bar{\lambda}$. This shows that λ is a real number. Now suppose that λ_1 and λ_2 are distinct eigenvalues with eigenvectors x_1 and x_2. Observe that, because L is selfadjoint and the eigenvalues are real, we have $\lambda_1\langle x_1, x_2\rangle = \langle Lx_1, x_2\rangle = \langle x_1, Lx_2\rangle = \lambda_2\langle x_1, x_2\rangle$. Thus, $(\lambda_1 - \lambda_2)\langle x_1, x_2\rangle = 0$ Since $\lambda_1 \neq \lambda_2$, dividing by $\lambda_1 - \lambda_2$ yields $\langle x_1, x_2\rangle = 0$. ∎

Before we go on to our next result, we need to set down a few facts about eigenvalues and eigenvectors, and about bases in general. This we now do.

Lemma 2.2. *Let V be a complex, finite dimensional vector space, with dimension $n \geq 1$. If $L : V \to V$ is linear, then L has at least one eigenvalue.*

[1] $\delta(x)$ plays the role of a Dirac delta function for the polynomial space.

Proof. Let B be a basis for V and let A_L be the matrix of L relative to B-coördinates. Because $\det(A_L - \lambda I) = 0$ is a polynomial in λ, it has n roots, if one counts repetitions. In any case, it has between 1 and n distinct roots, all of which are eigenvalues. Thus, L has at least one eigenvalue. ∎

Let us again return to our discussion of self-adjoint linear transformations. This time we will look at the case in which the underlying complex vector space is finite dimensional. In this case, self-adjoint transformations are always diagonalizable. Indeed, we can say even more, as the following result shows.

Theorem 2.3. *Let V be a complex, finite dimensional vector space. If $L : V \to V$ is a self-adjoint linear transformation, then there is an orthonormal basis for V that is composed of eigenvectors of L. The matrix of L relative to this basis is diagonal.*

We will give two proofs for this important theorem. The first is similar to the one given in Keener's book and involves invariant subspaces. The second is one that is more concrete in that it directly uses matrix computations. Here is the first proof.

Proof. (Proof 1). A subspace U of V is said to be invariant under $L : V \to V$ if and only if $L : U \to U$. Let $S := \text{span}\{\text{eigenvectors of } L\}$ and let $U = S^{\perp}$. We claim that $S^{\perp}$ is invariant under L. To see this, let $u \in U$ and $v_j \in S$ be an eigenvector of L. Since $\langle Lu, v_j\rangle = \langle u, Lv_j\rangle = \langle u, \lambda_j v_j\rangle$ and $u \in U = S^{\perp}$, we have $\langle Lu, v_j\rangle = \lambda_j \langle u, v_j\rangle = 0$. Thus, $Lu \in S^{\perp}$, so $S^{\perp}$ is invariant under L. If $S^{\perp} \neq \{0\}$, so that its dimension is one or more, then, by Lemma 2.2, L has an eigenvalue λ and with $v \neq 0$ being its corresponding eigenvector. But then v, being an eigenvector, is also in S. Thus, $v \in S \cap S^{\perp}$. It follows that $\langle v, v\rangle = 0$, and so $v = 0$. This is a contradiction, so $S^{\perp} = \{0\}$ and $V = S$. Using Gram-Schmidt if necessary, we may form an o.n. basis from the eigenvectors of L. ∎

Proof. (Proof 2). We will work with the matrix of L relative to an orthonormal basis. We denote this matrix by A. Of course this will be an $n \times n$ self adjoint matrix. By Lemma 2.2, A has an eigenvalue λ_1 with corresponding eigenvector x_1, normalized so that $\|x_1\| = 1$. Use Gram-Schmidt, if necessary, to form a orthonormal basis $B_1 = \{x_1, y_2, y_3, \ldots, y_n\}$. If we change coordinates to B_1, it is easy to show that A, in the new coordinates, becomes

$$A_1 = \begin{pmatrix} \lambda_1 & 0_{n-1}^T \\ 0_{n-1} & \tilde{A}_1 \end{pmatrix}$$

where $\widetilde{A}_1$ is a self adjoint $(n-1)\times(n-1)$ matrix. Repeating the argument for $\widetilde{A}_1$, we see that the matrix for $\widetilde{A}_1$ becomes A_2, where

$$A_2 = \begin{pmatrix} \lambda_1 & 0 & 0_{n-2}^T \\ 0 & \lambda_2 & 0_{n-2}^T \\ 0_{n-2} & 0_{n-2} & \widetilde{A}_2 \end{pmatrix}$$

Again $\widetilde{A}_2$ is self adjoint and put in a form similar to those above. Continuing in this way, we can find an orthonormal system of coordinates in $\mathbb{C}^n$ relative to which the matrix A is diagonal. The corresponding basis for V is also orthonormal and is composed eigenvectors of L. ∎

Of course, for a self-adjoint matrix A, Theorem 2.3 implies that there is a matrix $S = [x_1 \ \cdots x_n]$, whose columns are an o.n. set of eigenvectors of A, such that $A = S\Lambda S^{-1}$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. – note that he eigenvalues are listed in the same order as the eigenvectors. Since the columns of S are an o.n. set, it is easy to show that $S^{-1} = S^*$. In this form. we have

$$A = S\Lambda S^*, \ \ S^*S = I. \tag{3}$$

3 Applications

There are many important applications of what was discussed in the previous sections. The theory of adjoints and of self-adjoint linear transformations comes up in the study of partial differential equations and the eigenvalue problems that result when the method of separation of variables is used to solve them. (Partial differential equations arise in connection with heat conduction, wave propagation, fluid flow, electromagnetic fields, quantum mechanics, and many other areas as well.) Self-adjoint linear transformations play a fundamental role in formulating quantum mechanics; they represent the physical quantities that can be observed in a laboratory—the observables of physics. In this section we will provide a few examples.

Example 3.1. *Normal modes for a coupled spring system.* In the coupled spring system shown in Fig. 1, let $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. Using Hooke's law and Newton's law, the equation of motion for the spring system is, in matrix, form

$$\ddot{\mathbf{x}} = -\frac{k}{m}A\mathbf{x}, \quad A := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{4}$$

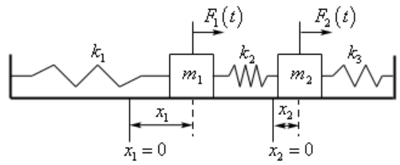

Figure 1: Coupled spring system

A normal mode for the system is a solution of the form $\mathbf{x}(t) = \text{function}(t)\mathbf{x}_0$, where $\mathbf{x}_0$ is independent of t. The usual way to treat this problem is to make the *Ansatz* $\mathbf{x}(t) = e^{i\omega t}\mathbf{x}_0$, where ω is a constant angular frequency. Plugging this solution into (4) and canceling the time factor, we obtain

$$\tfrac{m\omega^2}{k}\mathbf{x}_0 = A\mathbf{x}_0.$$

It follows that $m\omega^2/k = \lambda$ is an eigenvalue of A, with $\mathbf{x}_0$ being the corresponding eigenvector. In this case, we have two eigenvalues $\lambda = 1$, $\mathbf{x}_1 = (1\ 1)^T$ and $\lambda = 3$, $\mathbf{x}_3 = (1\ -1)^T$. The *eigenfrequencies* are thus $\omega_1 = \sqrt{k/m}$ and $\omega_3 = \sqrt{3k/m}$ and the corresponding normal modes are

$$e^{\pm i\sqrt{k/m}\,t}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad e^{\pm i\sqrt{3k/m}\,t}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Example 3.2. *Inertia tensor.* The kinetic energy of a rigid body freely rotating about its center of mass is

$$T = \frac{1}{2}\omega^T \mathcal{I}\omega.$$

The vector ω is the angular velocity of the body and $\mathcal{I}$ is the 3×3 inertia tensor. If $\rho(\mathbf{x})$ is the mass density of the body, which occupies the region $\Omega \subset \mathbb{R}^3$, then

$$\mathcal{I} = \int_\Omega \left(|\mathbf{x}|^2 I_{3\times 3} - \mathbf{x}\mathbf{x}^T\right)\rho(\mathbf{x})d^3\mathbf{x}.$$

The eigenvalues and eigenvectors of $\mathcal{I}$ play an important role in the equation of motion for a rigid body. (For details, see: H. Goldstein, *Classical Mechanics*, Addison-Wesley, 1965.)

Example 3.3. *Principal axis theorem.* Consider the conic $3x^2 - 2xy + 3y^2 = 1$. We want to rotate axes to find new coordinates x', y' relative to which the conic is in standard form. Let's put the equation in matrix form:

$$(x \quad y)\underbrace{\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

It is straightforward to show that the eigenvalues of A are 2 and 4, with corresponding orthonormal eigenvectors $\frac{1}{\sqrt{2}}(1\ 1)^T$ and $\frac{1}{\sqrt{2}}(-1\ 1)^T$. In the factored form in (3), we have

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

Let $\mathbf{x} = (x\ y)^T$. The original form of the conic was $\mathbf{x}^T A\mathbf{x} = 1$. If we set $\mathbf{x}' = S^T\mathbf{x}$, then the equation of the conic becomes $\mathbf{x}'^T\Lambda\mathbf{x}' = 1$ or, in the new coordinates, $2x'^2 + 4y'^2 = 1$. The matrix S is actually changes from x'-y' to x-y coordinates. In effect, it gets the x'-y' axes by rotating the x-y axes counter clockwise through an angle $\pi/4$.

4 The Courant-Fischer Theorem

It is simple to calculate the eigenvalues for small matrices, with $n = 2$ or 3. However, direct calculation is not possible for large systems. Thus, we need a method for approximating them. This is supplied by the Courant-Fischer Theorem, which we now state.

Theorem 4.1 (Courant-Fischer)**.** *Let A be a real $n \times n$ self-adjoint matrix having eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then,*

$$\lambda_k = \min_{C\in\mathbb{R}^{k-1\times n}} \max_{\substack{\|\mathbf{x}\|=1 \\ C\mathbf{x}=\mathbf{0}}} \mathbf{x}^T A\mathbf{x}. \tag{5}$$

Proof. Use (3) to get $\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T\Lambda\mathbf{y}$, where $\mathbf{y} = S^T\mathbf{x}$. Because S is orthogonal, we have $\|\mathbf{x}\| = \|S\mathbf{y}\| = \|\mathbf{y}\|$. In addition, CS runs over all matrices in $\mathbb{R}^{k-1\times n}$ if $C \in \mathbb{R}^{k-1\times n}$ does. Thus, we are now trying to show that

$$\lambda_k = \min_{C\in\mathbb{R}^{k-1\times n}} \max_{\substack{\|\mathbf{y}\|=1 \\ C\mathbf{y}=\mathbf{0}}} \mathbf{y}^T \Lambda\mathbf{y}. \tag{6}$$

Let $q(\mathbf{y}) = \mathbf{y}^T\Lambda\mathbf{y}$. Of course, q can be written as

$$q(\mathbf{y}) = \sum_{j=1}^{n} \lambda_j y_j^2.$$

The proof proceeds in two steps. First, to satisfy $C_0\mathbf{y} = \mathbf{0}$ when $C_0 = [e_1 \ \cdots \ e_{k-1}]^T$, we need only take $\mathbf{y} = \sum_{j=k}^n y_j e_j$. In that case, we have, since $\|\mathbf{y}\|^2 = \sum_{j=k}^n y_j^2 = 1$,

$$q(\mathbf{y}) = \sum_{j=k}^{n} \lambda_j y_j^2 \leq \lambda_k \sum_{j=k}^{n} y_j^2 = \lambda_k \cdot 1 = \lambda_k,$$

and so, for C_0, we have $\max_{\substack{\|\mathbf{y}\|=1 \\ C_0\mathbf{y}=\mathbf{0}}} q(\mathbf{y}) = \lambda_k$.

The second step is to show that for any C we can find a $\mathbf{y}'$ such that $C\mathbf{y}' = 0$ and $q(\mathbf{y}') \geq \lambda_k$. If we can do that, then

$$\max_{\substack{\|\mathbf{y}\|=1 \\ C\mathbf{y}=\mathbf{0}}} q(\mathbf{y}) \geq q(\mathbf{y}') \geq \lambda_k = \max_{\substack{\|\mathbf{y}\|=1 \\ C_0\mathbf{y}=\mathbf{0}}} q(\mathbf{y}),$$

and (6) follows immediately. To show that such a $\mathbf{y}$ exists, start by augmenting C by adding rows e_j^T, $j = k+1, \ldots n$:

$$\widetilde{C} = \begin{pmatrix} C \\ e_{k+1}^T \\ \vdots \\ e_n^T \end{pmatrix} \in \mathbb{R}^{(n-1)\times n}.$$

Note that since $\operatorname{rank}(\widetilde{C}) \leq n-1$, so $\operatorname{nullity}(\widetilde{C}) \geq 1$. Thus there is a vector $\mathbf{y}' \neq \mathbf{0}$ such that $\widetilde{C}\mathbf{y}' = \mathbf{0}$. This is equivalent to the equations $C\mathbf{y}' = \mathbf{0}$ and $y_j' = e_j^T\mathbf{y}' = 0$, $j = k+1, \ldots, n$. Moreover, this implies that

$$q(\mathbf{y}') = \sum_{j=1}^{k} \lambda_j {y_j'}^2 \geq \lambda_k \sum_{j=1}^{k} {y_j'}^2 = \lambda_k \cdot 1 = \lambda_k.$$

This completes the proof. ∎

Example 4.2. *Estimating an eigenvalue.* Show that $\lambda_2 \leq 0$, for the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{pmatrix}.$$

Because A has positive entries, we expect that the eigenvector for λ_1 will have all positive entries. (This is in fact a consequence of the Perron-Frobenius Theorem.) Thus, since we want to get an estimate of the minimum of the maximum of $\mathbf{x}^T A\mathbf{x}$, we guess that $C = (1\ 1\ 1)$ and so $C\mathbf{x} =$

$x_1 + x_2 + x_3 = 0$. The quadratic form

$$q(\mathbf{x}) := \mathbf{x}^T A\mathbf{x} = x_1x_2 + 2x_1x_3 + 3x_2x_3.$$

Let's solve $x_1+x_2+x_3 = 0$ for x_1 and put the result in the expression above for $q(\mathbf{x})$. Doing so yields

$$q(\mathbf{x}) = -(x_2 + x_3)(x_2 + 2x_3) + +3x_2x_3 = -x_2^2 - 2x_3^2 \le 0. \qquad (7)$$

From this, we see that for $C = (1\ 1\ 1)$ and $C\mathbf{x} = 0$, $\lambda_2 \le \max_{C\mathbf{x}=0} q(\mathbf{x}) \le 0$. If we had $\lambda_2 = 0$, then the inequality (7) would imply that $-x_2^2 - 2x_3^2 = 0$, so we would have $x_2 = x_3 = 0$. But $x_1 + x_2 + x_3 = 0$, which implies that $x_1 = 0$, too. This is impossible because $x_1^2 + x_2^2 + x_3^2 = 1$.

5 The Fredholm Alternative

We now turn to a discussion of when we can solve $Lv = w$, for $L : V \to W$ and linear. Both V and W are assumed to be real or complex, finite dimensional inner product spaces; the dimension of V being n and that of W is m. We will allow $m \ne n$. We wish to prove the following theorem.

Theorem 5.1 (The Fredholm Alternative)**.** *The equation $Lv = w$ has a solution if and only if $w \in \mathrm{Null}(L^*)^\perp$ –i.e., $\mathrm{Range}(L) = \mathrm{Null}(L^*)^\perp$. Equivalently, either $\langle w, u\rangle_W = 0$ for all $u \in \mathrm{Null}(L^*)$, in which case $Lv = w$ has a solution, or there is some $u \in \mathrm{Null}(L^*)$ such that $\langle w, u\rangle_W \ne 0$, in which case $Lv = w$ does not have a solution.*

Proof. Suppose that there is a $v \in V$ such that $Lv = w$. Let $u \in \mathrm{Null}(L^*)$. Then, $\langle w, u\rangle_W = \langle Lv, u\rangle_W = \langle v, L^*u\rangle_V = \langle v, 0\rangle_V = 0$. Hence, $w \in \mathrm{Null}(L^*)^\perp$, and so $\mathrm{Range}(L) \subseteq \mathrm{Null}(L^*)^\perp$. We now need to show that $\mathrm{Null}(L^*)^\perp = \mathrm{Range}(L)$.

Suppose this is false, so that there is some $w \in \mathrm{Null}(L^*)^\perp$ that is not in $\mathrm{Range}(L)$. We may also assume that $w \in \mathrm{Range}(L)^\perp$. To see this, note that P, the orthogonal projection of w onto $\mathrm{Range}(L)$, exists, because all of the spaces involved are finite dimensional. Consequently, $w = (I - P)w + Pw$. Because w is *not* in the range of L, $(I - P)w \ne 0$ and is also in $\mathrm{Range}(L)^\perp$. Just replace $(I - P)w$ by w to validate the assumption that $w \in \mathrm{Range}(L)^\perp \cap \mathrm{Null}(L^*)^\perp$. Note that if $v \in V$, than $Lv \in \mathrm{Range}(L)$, so $\langle Lv, w\rangle_W = 0 = \langle v, L^*w\rangle_V$. Choose $v = L^*w$, which is in V. Doing so gives us $\langle L^*w, L^*w\rangle_V = 0$ and, consequently $L^*w = 0$, so $w \in \mathrm{Null}(L^*)$. We also have that $w \in \mathrm{Null}(L^*)^\perp$. Since $\mathrm{Null}(L^*) \cap \mathrm{Null}(L^*)^\perp = \{0\}$, $w = 0$. This is a contradiction. It follows that $\mathrm{Range}(L) = \mathrm{Null}(L^*)^\perp$. ∎

Remark 5.2. Later on, we will see that this result holds in a Hilbert space setting. The only place that we used the finite dimensionality of the spaces involved was in showing the existence of the orthogonal projection P.

Example 5.3. Let $k(x, y) = x + 3x^2y + xy^2$ and $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$. Consider the operator $Lu = \int_{-1}^{1} k(x, y)u(y)dy$.

1. Show that $L : P_2 \to P_2$.
2. Find L^* and $\mathrm{Null}(L^*)$.
3. Find a condition on $q \in P_2$ for which $Lp = q$ always has a solution.

Solution. (1) We have $Lp = x\langle p, 1\rangle + 3x^2\langle p, y\rangle + x\langle p, y^2\rangle$, which is in P_2. Thus, $L : P_2 \to P_2$. (2) It is easy to see that $L^*p = \int_{-1}^{1} k(y, x)p(y)dy$, and so $L^*p = \int_{-1}^{1} \big(y + 3y^2x + x^2y\big)p(y)dy = \langle p, y\rangle + 3x\langle p, y^2\rangle + x^2\langle p, y\rangle$. Furthermore, if $p \in \mathrm{Null}(L^*)$, then, since $1, x, x^2$ are linearly independent, p satisfies $\langle p, y\rangle = 0$ and $\langle p, y^2\rangle = 0$. Let $p(x) = a_0 + a_1x + a_2x^2$. Note that $\langle p, y\rangle = 2a_1/3$ and $\langle p, y^2\rangle = 2a_0/3 + 2a_2/5 = 0$. It follows that $a_1 = 0$ and $a_2 = -5a_0/3$, so $p(x) = a_0(1 - 5x^2/3)$. Consequently, $\mathrm{Null}(L^*) = \mathrm{span}\{1 - 5x^2/3\}$. (3) By the Fredholm Alternative, the condition on q is then that $\langle q(x), 1 - 5x^2/3\rangle = 0$.

Previous: coordinates and bases
Next: Banach spaces and Hilbert spaces