**Several Important Theorems**
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# 1 The Projection Theorem

Let $\mathcal{H}$ be a Hilbert space. When $V$ is a finite dimensional subspace of $\mathcal{H}$ and $f \in \mathcal{H}$, we can always find a unique $p \in V$ such that $\|f - p\| = \min_{v\in V} \|f - v\|$. This fact is the foundation of least-squares approximation. What happens when we allow $V$ to be infinite dimensional? We will see that the minimization problem can be solved if and only if $V$ is closed.

**Theorem 1.1** (The Projection Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space and let $V$ be a subspace of $\mathcal{H}$. For every $f \in \mathcal{H}$ there is a unique $p \in V$ such that $\|f - p\| = \min_{v\in V} \|f - v\|$ if and only if $V$ is a closed subspace of $\mathcal{H}$.*

To prove this, we need the following lemma.

**Lemma 1.2** (Polarization Identity)**.** *Let $\mathcal{H}$ be a Hilbert space. For every pair $f, g \in \mathcal{H}$, we have*

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2).$$

*Proof.* Adding the $\pm$ identities $\|f \pm g\|^2 = \|f\|^2 \pm \langle f, g\rangle \pm \langle g, f\rangle + \|g\|^2$ yields the result. □

The polarization identity is an easy consequence of having an inner product. It is surprising that if a *norm* satisfies the polarization identity, then the norm *comes* from an inner product[1].

*Proof.* (Projection Theorem) Showing that the existence of minimizer implies that $V$ is closed is left as an exercise. So we assume that $V$ is closed. For $f \in \mathcal{H}$, let $\alpha := \inf_{v\in V} \|v - f\|$. It is a little easier to work with this in an equivalent form, $\alpha^2 = \inf_{v\in V} \|v - f\|^2$. Thus, for every $\varepsilon > 0$ there is a $v_\varepsilon \in V$ such that $\alpha^2 \le \|v_\varepsilon - f\|^2 < \alpha^2 + \varepsilon$. By choosing $\varepsilon = 1/n$, where $n$ is a positive integer, we can find a sequence $\{v_n\}_{n=1}^\infty$ in $V$ such that

$$0 \le \|v_n - f\|^2 - \alpha^2 < \frac{1}{n} \tag{1.1}$$

[1] Jordan, P. ; Von Neumann, J. On inner products in linear, metric spaces. Ann. of Math. (2) 36 (1935), no. 3, 719–723.

Of course, the same inequality holds for a possibly different integer $m$, $0 \le \|v_m - f\|^2 - \alpha^2 < \frac{1}{m}$. Adding the two yields this:

$$0 \le \|v_n - f\|^2 + \|v_m - f\|^2 - 2\alpha^2 < \frac{1}{n} + \frac{1}{m}. \tag{1.2}$$

By polarization identity and a simple manipulation, we have

$$\|v_n - v_m\|^2 + 4\|f - \frac{v_n + v_m}{2}\|^2 = 2\big(\|f - v_n\|^2 + \|f - v_m\|^2\big).$$

We can subtract $4\alpha^2$ from both sides and use (1.2) to get

$$\|v_n - v_m\|^2 + 4(\|f - \frac{v_n + v_m}{2}\|^2 - \alpha^2) = 2(\|f - v_n\|^2 + \|f - v_m\|^2 - 2\alpha^2) < \frac{2}{n} + \frac{2}{m}.$$

Because $\frac{1}{2}(v_n + v_m) \in V$, $\|f - \frac{v_n + v_m}{2}\|^2 \ge \inf_{v\in V} \|v - f\|^2 = \alpha^2$. It follows that the second term on the left is nonnegative. Dropping it makes the left side smaller:

$$\|v_n - v_m\|^2 < \frac{2}{n} + \frac{2}{m}. \tag{1.3}$$

As $n, m \to \infty$, we see that $\|v_n - v_m\| \to 0$. Thus $\{v_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{H}$ and is therefore convergent to a vector $p \in \mathcal{H}$. Since $V$ is closed, $p \in V$. Furthermore, taking limits in (1.1) implies that $\|p - f\| = \inf_{v\in V} \|v - f\|$. The uniqueness of $p$ is left as an exercise. $\square$

There are two important corollaries to this theorem; they follow from problem 4 of Assignment 1, 2021, and Theorem 1.1. We list them below.

**Corollary 1.3.** *Let $V$ be a subspace of $\mathcal{H}$. There exists an orthogonal projection $P : \mathcal{H} \to V$ for which $\|f - Pf\| = \min_{v\in V} \|f - v\|$ if and only if $V$ is closed.*

**Corollary 1.4.** *Let $V$ be a closed subspace of $\mathcal{H}$. Then, $\mathcal{H} = V \oplus V^\perp$ and $(V^\perp)^\perp = V$.*

## 2 The Riesz Representation Theorem

Let $V$ be a Banach space. A bounded linear transformation $\Phi$ that maps $V$ into $\mathbb{R}$ or $\mathbb{C}$ is called a *linear functional* on $V$. The linear functionals form a Banach space $V^*$, called the *dual space* of $V$, with norm defined by

$$\|\Phi\|_{V^*} := \sup_{v\ne 0} \frac{|\Phi(v)|}{\|v\|_V}\,.$$

## 2.1 The linear functionals on Hilbert space

**Theorem 2.1** (The Riesz Representation Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space and let $\Phi : \mathcal{H} \to \mathbb{C}$ (or $\mathbb{R}$) be a bounded linear functional on $\mathcal{H}$. Then, there is a unique $g \in \mathcal{H}$ such that, for all $f \in \mathcal{H}$, $\Phi(f) = \langle f, g \rangle$.*

*Proof.* The functional $\Phi$ is a bounded operator that maps $\mathcal{H}$ into the scalars. It follows from our discussion of bounded operators that the null space of $\Phi$, $N(\Phi)$, is closed. If $N(\Phi) = \mathcal{H}$, then $\Phi(f) = 0$ for all $f \in \mathcal{H}$, hence $\Phi = 0$. Thus we may take $g = 0$. If $N(\Phi) \neq \mathcal{H}$, then, since $N(\Phi)$ is closed, we have that $\mathcal{H} = N(\Phi) \oplus N(\Phi)^\perp$. In addition, since $N(\Phi) \neq \mathcal{H}$, there exists a nonzero vector $h \in N(\Phi)^\perp$. Moreover, $\Phi(h) \neq 0$, because $h$ is not in the null space $N(\Phi)$. Next, note that for $f \in \mathcal{H}$, the vector $w := \Phi(h)f - \Phi(f)h$ is in $N(\Phi)$. To see this, observe that

$$\Phi(w) = \Phi\big(\Phi(h)f - \Phi(f)h\big) = \Phi(h)\Phi(f) - \Phi(f)\Phi(h) = 0.$$

Because $w = \Phi(h)f - \Phi(f)h \in N(\Phi)$, it is orthogonal to $h \in N(\Phi)^\perp$, we have that

$$0 = \langle \Phi(h)f - \Phi(f)h, h \rangle = \Phi(h)\langle f, h \rangle - \Phi(f) \underbrace{\langle h, h \rangle}_{\|h\|^2}.$$

Solving this equation for $\Phi(f)$ yields $\Phi(f) = \langle f, \frac{\overline{\Phi(h)}}{\|h\|^2} h \rangle$. The vector $g := \frac{\overline{\Phi(h)}}{\|h\|^2} h$ then satisfies $\Phi(f) = \langle f, g \rangle$. To show uniqueness, suppose $g_1, g_2 \in \mathcal{H}$ satisfy $\Phi(f) = \langle f, g_1 \rangle$ and $\Phi(f) = \langle f, g_2 \rangle$. Subtracting these two gives $\langle f, g_2 - g_1 \rangle = 0$ for all $f \in \mathcal{H}$. Letting $f = g_2 - g_1$ results in $\langle g_2 - g_1, g_2 - g_1 \rangle = 0$. Consequently, $g_2 = g_1$. $\square$

## 2.2 Adjoints of bounded linear operators

We now turn the problem of showing that an adjoint for a bounded operator always exists. This is just a corollary of the Riesz Representation Theorem.

**Corollary 2.2.** *Let $L : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Then there exists a bounded linear operator $L^* : \mathcal{H} \to \mathcal{H}$, called the adjoint of $L$, such that $\langle Lf, h \rangle = \langle f, L^* h \rangle$, for all $f, h \in \mathcal{H}$.*

*Proof.* Fix $h \in \mathcal{H}$ and define the linear functional $\Phi_h(f) = \langle Lf, h \rangle$. Using the boundedness of $L$ and Schwarz's inequality, we have $|\Phi_h(f)| \leq \|L\|\|f\|\|h\| = K\|f\|$, and so $\Phi_h$ is bounded. By Theorem 2.1, there is a unique vector $g$ in $\mathcal{H}$ for which $\Phi_h(f) = \langle f, g \rangle$. The vector $g$ is uniquely determined by $\Phi_h$; thus $g = g_h$ a function of $h$. We claim that $g_h$ is a linear function of $h$. Consider $h = ah_1 + bh_2$. Note that $\Phi_h(f) = \langle Lf, ah_1 + bh_2 \rangle = \bar{a}\Phi_{h_1}(f) + \bar{b}\Phi_{h_2}(f)$. Since $\Phi_{h_1}(f) = \langle f, g_1 \rangle$ and $\Phi_{h_2}(f) = \langle f, g_2 \rangle$, we see that

$$\Phi_h(f) = \langle f, g_h \rangle = \bar{a}\Phi_{h_2}(f) + \bar{b}\Phi_{h_2}(f) = \langle f, ag_{h_1} + bg_{h_2} \rangle.$$

It follows that $g_h = ag_{h_1} + bg_{h_2}$ and that $g_h$ is a linear function of $h$. It is also bounded. If $f = g_h$, then $\Phi_h(g_h) = \|g_h\|^2$. From the bound $|\Phi_h(f)| \le \|L\|\|f\|\|h\|$, we have $\|g_h\|^2 \le |L\|\|g_h\|\|h\|$. Dividing by $\|g_h\|$ then yields $\|g_h\| \le \|L\|\|h\|$. Thus the correspondence $h \to g_h$ is a bounded linear function on $\mathcal{H}$. Denote this function by $L^*$. Since $\langle Lf, h \rangle = \langle f, g_h \rangle$, we have that $\langle Lf, h \rangle = \langle f, L^*h \rangle$. □

**Corollary 2.3.** $\|L^*\| = \|L\|$.

*Proof.* By problem 7 in Assignment 7, 2021, $\|L\| = \sup_{f,h} |\langle Lf, h \rangle|$, where $\|h\| = \|f\| = 1$. On the other hand, $\|L^*\| = \sup_{f,h} |\langle L^*h, f \rangle|$. Since $\langle L^*h, f \rangle = \overline{\langle f, L^*h \rangle}$, we have that $\sup_{f,h} |\langle L^*h, f \rangle| = \sup_{f,h} |\langle Lf, h \rangle|$. It immediately follows that $\|L^*\| = \|L\|$. □

**Example 2.4.** *Let $R = [0,1] \times [0,1]$ and suppose that $k$ is a Hilbert-Schmidt kernel. If $Lu(x) = \int_0^1 k(x,y)u(y)dy$, then $L^*v(x) = \int_0^1 \overline{k(y,x)}v(y)dy$.*

*Proof.* We will use $s, t$ as the integration variables and switch back, to avoid confusion. We begin with $\langle Lu, v \rangle = \int_0^1 \left( \int_0^1 k(s,t)u(t)dt \right) \overline{v(s)}ds$. By Fubini's theorem, we may switch the variables of integration to get this:

$$\begin{aligned} \int_0^1 \left( \int_0^1 k(s,t)u(t)dy \right) \overline{v(s)}ds &= \int_0^1 \left( \int_0^1 k(s,t)\overline{v(s)}ds \right) u(t)dt \\ &= \int_0^1 \left( \underbrace{\overline{\int_0^1 \overline{k(s,t)}v(s)ds}}_{L^*v} \right) u(t)dt. \\ &= \langle u, L^*v \rangle \end{aligned}$$

The result follows by changing variables from $t, s$ to $x, y$ in the second equation above . □

## 3 The Fredholm Alternative

**Theorem 3.1** (The Fredholm Alternative)**.** *Let $L : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator whose range, $R(L)$, is closed. Then, the equation $Lf = g$*

*and be solved if and only if* $\langle g, v\rangle = 0$ *for all* $v \in N(L^*)$*. Equivalently,* $R(L) = N(L^*)^\perp$*.*

*Proof.* Let $g \in R(L)$, so that there is an $h \in \mathcal{H}$ such that $g = Lh$. If $v \in N(L^*)$, then $\langle g, v\rangle = \langle Lh, v\rangle = \langle h, L^*v\rangle = 0$. Consequently, $R(L) \subseteq N(L^*)^\perp$. Let $f \in N(L^*)^\perp$. Since $R(L)$ is closed, the projection theorem, Theorem 1.1, and Corollary 1.3, imply that there exists an orthogonal projection $P$ onto $R(L)$ such that $Pf \in R(L)$ and $f' = f - Pf \in R(L)^\perp$. Moreover, since $f$ and $Pf$ are both in $N(L^*)^\perp$, we have that $f' \in R(L)^\perp \cap N(L^*)^\perp$. Hence, $\langle Lh, f'\rangle = 0 = \langle h, L^*f'\rangle$, for all $h \in \mathcal{H}$. Setting $h = L^*f'$ then yields $L^*f' = 0$, so $f' \in N(L^*)$. But $f' \in N(L^*)^\perp$ and is thus orthogonal to itself; hence, $f' = 0$ and $f = Pf \in R(L)$. It immediately follows that $N(L^*)^\perp \subseteq R(L)$. Since we already know that $R(L) \subseteq N(L^*)^\perp$, we have $R(L) = N(L^*)^\perp$. $\square$

We want to point out that $R(L)$ being closed is crucial for the theorem to be true. If it is not closed, then the projection $P$ will not exist and the proof breaks down. In that case, one actually has $\overline{R(L)} = N(L^*)^\perp$, but *not* $R(L) = N(L^*)^\perp$.

The theorem is stated in a variety of ways. The form that emphasizes the "alternative" is given in the result below, which follows immediately from Theorem 3.1.

**Corollary 3.2.** *Let* $L : \mathcal{H} \to \mathcal{H}$ *be a bounded linear operator whose range,* $R(L)$*, is closed. Then, either the equation* $Lf = g$ *has a solution or there exists a vector* $v \in N(L^*)$ *such that* $\langle g, v\rangle \neq 0$*.*

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