Spectral Theory for Compact Self–Adjoint Operators
by
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Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ denote the bounded linear operators on $\mathcal{H}$ and the compact operators on $\mathcal{H}$, respectively.

1 The Resolvent Set and the Spectrum of an Operator

For $n \times n$ matrices, the spectrum is just the set of eigenvalues. The spectrum of a linear operator L is defined indirectly, as the complement of another set, the resolvent set. It is necessary to do this because on an infinite dimensional space the operator L may not have eigenvalues in the usual sense.

Definition 1.1. *Let $L \in \mathcal{B}(\mathcal{H})$. The* resolvent set *of L is $\rho(L) := \{\lambda \in \mathbb{C} \colon (L - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H})\}$*[2]*. The operator $R_L(\lambda) := (L - \lambda I)^{-1}$ is called the* resolvent *of L. The* spectrum *of L, $\sigma(L)$, is defined as the* complement *of the resolvent set: $\sigma(L) := \rho(L)^{\complement}$.*

This agrees with the definition of the spectrum in the matrix case, where the resolvent set comprises all complex numbers that are *not* eigenvalues. In terms of its spectrum, we will see that a compact operator behaves like a matrix, in the sense that its spectrum is the union of all of its eigenvalues and 0. We begin with the eigenspaces of a compact operator.

We start with two lemmas that we will need in the sequel. The first holds for all self-adjoint operators, including unbounded ones.

Lemma 1.2. *Let $L = L^*$ be in $\mathcal{B}(\mathcal{H})$. Then the eigenvalues of L are real and the eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Proof. The proof is identical to the one given in the matrix case, and so we will skip it. □

The second lemma, which we proved earlier, is used throughout this section. In particular, it is used in the three propositions following it.

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[2] $(L - \lambda I)^{-1}$ may exist as an unbounded operator, but, for λ to be in the resolvent set, this inverse must be bounded.

Lemma 1.3. *Let $\{\phi_n\}_{n=1}^{\infty}$ be an o.n. set in $\mathcal{H}$ and let $K \in \mathcal{C}(\mathcal{H})$. Then, $\lim_{n\to\infty} K\phi_n = 0$.*

Proof. See Lemma 2.4, Compact Sets and Compact Operators. □

Proposition 1.4. *If $K \in \mathcal{C}(\mathcal{H})$, then $\sigma(K)$ consists only of eigenvalues of K, together with 0.*

Proof. We will assume that $K = K^*$. The result *is* true for all compact operators, but the proof for the general case requires more work[3]. Suppose that $\lambda \in \sigma(K)$, $\lambda \neq 0$. By definition, $K - \lambda I$ is not boundedly invertible. This can happen either because there is a vector $u \in \mathcal{H}$, $u \neq 0$, such that $Ku = \lambda u$, or the range of $K - \lambda I$ is *not* all of $\mathcal{H}$, or both. If the former holds, then λ is an eigenvalue of K and we are done. So, we will suppose that the range of $K - \lambda I$ is *not* all of $\mathcal{H}$. Because K is compact, the Fredholm alternative applies to the operator[4] $L = K - \lambda I$. Thus, $\mathcal{H} = N(L^*) \oplus R(L)$. Since, by assumption, $R(L) \neq \mathcal{H}$, there is a least one $w \in N(L^*)$, $w \neq 0$; that is, $L^*w = K^*w - \bar{\lambda}w = 0$. But $K^* = K$ and thus $Kw = \bar{\lambda}w$, which means that $\bar{\lambda}$ is an eigenvalue of K. However, all of the eigenvalues of $K = K^*$ are real. Thus $\bar{\lambda} = \lambda$, and hence λ is itself an eigenvalue.

We now turn to showing that 0 is in $\sigma(K)$. Suppose not. Then, $0 \in \rho(K)$ and $K^{-1} \in \mathcal{B}(\mathcal{H})$. Let $\{\phi_n\}_{n=1}^{\infty}$ be an o.n. set and let $\psi_n = K\phi_n$. Then, since $\phi_n = K^{-1}\psi_n$, we have that $\|\phi_n\| = 1 \leq \|K^{-1}\|\|\psi_n\|$. But, by Lemma 1.3, we have that $\lim_{n\to\infty} \|K\phi_n\| = \lim_{n\to\infty} \|\psi_n\| = 0$, which is a contradiction. Thus, 0 is in $\sigma(K)$. □

Proposition 1.5. *Let $K \in \mathcal{C}(\mathcal{H})$. If $\lambda \neq 0$ is an eigenvalue of K, with corresponding eigenspace $\mathcal{E}_\lambda$, then $\mathcal{E}_\lambda$ is finite dimensional.*

Proof. Because $\mathcal{E}_\lambda = N(K - \lambda I)$, the eigenspace is closed. We may therefore choose an o.n basis $\{\phi_n\}_{n=1}^{N}$ for $\mathcal{E}_\lambda$, using the Gram-Schmidt process if necessary. Suppose that $N = \infty$. Then we have that $K\phi_n = \lambda\phi_n$. Since the ϕ_n's are o.n., this implies that $\|K\phi_n\| = |\lambda| \neq 0$. But, by Lemma 1.3, we have that $\lim_{n\to\infty} \|K\phi_n\| = 0$. This contradiction implies that N is finite. □

Proposition 1.6. *Let $K \in \mathcal{C}(\mathcal{H})$ be self-adjoint. Then 0 is the only possible accumulation point of the eigenvalues of K*

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[3]See T. Kato, *Perturbation Theory for Linear Operators*, Theorem 6.26, p. 185.

[4]The notation used earlier was $L = I - \lambda K$. Because of the definitions of the spectrum and resolvent, this is inconvenient here.

Proof. Suppose not. Then we may choose a sequence of (distinct) eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \lambda_n = \lambda \neq 0$. Let ϕ_n be an eigenvector corresponding to λ_n, with $\|\phi_n\| = 1$. Because the eigenvalues are distinct, the set $\{\phi_n\}_{n=1}^{\infty}$ is orthonormal. As above, this implies two things: First, since $\|K\phi_n\| = |\lambda_n|$, $\lim_{n\to\infty} \|K\phi_n\| = \lim_{n\to\infty} |\lambda_n| = |\lambda|$. Second, by Lemma 1.3, $\lim_{n\to\infty} K\phi_n = 0$. Combining the two yields $\lambda = 0$, which is a contradiction. □

We remark that the previous proposition is true for any compact operator, not just ones that are self adjoint.

2 Spectral Theory for Self-Adjoint Compact Operators

In this section we will prove that the self-adjoint compact operators have properties very similar to self-adjoint matrices. Essentially, the difference comes in there being an infinite o.n. basis of for $\mathcal{H}$ composed of eigenvectors of the operator. This has application to eigenvalue problems associated with differential equations.

Lemma 2.1. *Let $L = L^*$ be in $\mathcal{B}(\mathcal{H})$. Then $\|L\| = \sup_{\|u\|=1} |\langle Lu, u\rangle|$.*

Proof. See problem 6(c), assignment 10. □

Lemma 2.2. *Let $K \neq 0 \in \mathcal{C}(\mathcal{H})$ be self-adjoint. Then, either $\|K\|$ or $-\|K\|$ or possibly both, are eigenvalues.*

Proof. By Lemma 2.1, $\|K\| = \sup_{\|u\|=1} |\langle Ku, u\rangle|$. Thus we can choose a sequence $\{u_n\}_{n=1}^{\infty}$, $\|u_n\| = 1$, such that $\|K\| = \lim_{n\to\infty} |\langle Ku_n, u_n\rangle|$. Taking away absolute values, we see that the sequence $\langle Ku_n, u_n\rangle$ will converge to $\|K\|$, or $-\|K\|$, or may have subsequences that converge to either of these. We will assume that $\langle Ku_n, u_n\rangle$ converges to $\|K\|$. If not, reverse the sign of K. Next, note that

$$\begin{aligned} \|Ku_n - \|K\|\, u_n\|^2 &= \|Ku_n\|^2 - 2\|K\|\langle Ku_n, u_n\rangle + \|K\|^2 \\ &\leq \|K\|^2 - 2\|K\|\langle Ku_n, u_n\rangle + \|K\|^2 \\ &\leq 2\|K\|\big(\|K\| - \langle Ku_n, u_n\rangle\big) \end{aligned} \tag{2.1}$$

Because $\|K\| = \lim_{n\to\infty}\langle Ku_n, u_n\rangle$, we have $\lim_{n\to\infty} \big\|Ku_n - \|K\|u_n\big\| = 0$. Note that his does *not* mean that the sequence $\{u_n\}$ is convergent, only that $z_n := Ku_n - \|K\|u_n$ converges to 0. In fact, this result applies for any self-adjoint operator $L^* = L \in \mathcal{B}(\mathcal{H})$, not just self-adjoint compact operators.

We will now make use of K being compact. Since the sequence $\{u_n\}$ satisfies $\|u_n\| = 1$, it is bounded, we can extract a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ for which $\{Ku_{n_k}\}$ is convergent. It follows that

$$\|K\| \lim_{k\to\infty} u_{n_k} = \lim_{k\to\infty} Ku_{n_k} - \lim_{k\to\infty} z_{n_k}.$$

Since $z_{n_k} \to 0$ and the $\lim_{k\to\infty} Ku_{n_k}$ exists, we have that $u := \lim_{k\to\infty} u_{n_k}$ exists as well, and, consequently, that $\lim_{k\to\infty} Ku_{n_k} = Ku$. Moreover, using this in the previous equation and noting that $z_{n_k} \to 0$, we have $\|K\|u = Ku + 0 = Ku$. Finally, because $\|u\| = \|\lim_{k\to\infty} u_{n_k}\| = 1$, $\|K\|$ is an eigenvalue of K, with $u \neq 0$ being an eigenvector. □

We can obtain all of the eigenvalues in the same way as we did above. In showing this, we will simplify the notation in the discussion by assuming that the operator $K = K^*$ satisfies $\langle Kv, v\rangle \geq 0$ for all $v \in \mathcal{H}$. An operator with this property is said to be *nonnegative*. This really doesn't change the argument we will now give. We will begin with the idea of an invariant subspace:

Definition 2.3. *We say that a subspace $\mathcal{U}$ of a Hilbert space $\mathcal{H}$ is invariant under an operator $L \in \mathcal{B}(\mathcal{H})$ if and only if for all $v \in \mathcal{U}$, Lv is in $\mathcal{U}$.*

Invariance will enable us to put a self-adjoint operator in "diagonal" form. To see what we mean, let $K^* = K \in \mathcal{C}(\mathcal{H})$. Label the first n positive eigenvalues in decreasing order, $\|K\| = \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$, and let M_n be the span of all of the eigenvectors corresponding to $\lambda_1, \ldots, \lambda_n$ and let $M_n^\perp$ be its orthogonal complement in $\mathcal{H}$. Then we have the result below.

Lemma 2.4. *Both M_n and $M_n^\perp$ are invariant under K.*

Proof. Any $v \in M_n$ is a linear combination of eigenvectors of K; i.e., $v = \sum_{j=1}^n \alpha_j u_j$, where $Ku_j = \lambda_j u_j$. Hence,

$$Kv = \sum_{j=1}^n \alpha_j K u_j = \sum_{j=1}^n \alpha_j \lambda_j u_j \in M_n,$$

so M_n is invariant under K. To see that $M_n^\perp$ is also invariant we must show that if $w \in M_n^\perp$, then $Kw \in M_n^\perp$. Let $v \in M_n$ and $w \in M_n^\perp$, so the invariance of M_n implies that $Kv \in M_n$ and, hence, $\langle Kv, w\rangle = 0$. However, since $K = K^*$,

$$0 = \langle Kv, w\rangle = \langle v, K^* w\rangle = \langle v, Kw\rangle,$$

which gives us that $\langle v, Kw\rangle = 0$ and also that $\langle Kw, v\rangle = 0$. It follows that $Kw \in M_n^\perp$ and so $M_n^\perp$ is invariant under K. □

Lemma 2.5. *Let $K \neq 0 \in \mathcal{C}(\mathcal{H})$ be self-adjoint and nonnegative. If K has n positive eigenvalues $\|K\| = \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$, then*

$$\lambda_{n+1} = \sup\{\langle Ku, u\rangle \colon u \in M_n^\perp,\ \|u\| = 1\} < \lambda_n, \tag{2.2}$$

where M_n is the span of all of the eigenvectors for λ_1 through λ_n.

Proof. The subspace M_n is invariant under K, and so is its orthogonal complement $M_n^\perp$. We now define K_{n+1} be the restriction of K to $M_n^\perp$: For all $w \in M_n^\perp$, $K_{n+1}w := Kw$. It is easy to see that K being compact on $\mathcal{H}$ implies that K_{n+1} is compact on $M_n^\perp$. By Lemma 2.2, with K_{n+1} replacing K and and $M_n^\perp$ replacing $\mathcal{H}$, we have that

$$\|K_{n+1}\| = \sup\{\langle K_{n+1}w, w\rangle \colon w \in M_n^\perp,\ \|u\| = 1\} \tag{2.3}$$

is an eigenvalue of K_{n+1}, with $w \neq 0$ being a corresponding eigenvector; that is, $K_{n+1}w = \|K_{n+1}\|w$. However, since K_{n+1} is the restriction of K to $M_n^\perp$, we see that $K_{n+1}w = Kw = \|K_{n+1}\|w$. Consequently, $\|K_{n+1}\|$ is an eigenvalue of K as well. Let $\lambda_{n+1} := \|K_{n+1}\|$. We leave it as an exercise to show that $\lambda_{n+1} < \lambda_n$. □

Proposition 2.6. *From among eigenvectors of K corresponding to the nonzero eigenvalues of K, one may select an orthonormal basis for $R(K)$. Moreover, if $R(K)$ is dense in $\mathcal{H}$, then that set forms an orthonormal basis for $\mathcal{H}$.*

Proof. We will use the notation introduced in Lemma 2.5. In addition, take P_n to be the orthogonal projection of $\mathcal{H}$ onto M_n and $P_n^\perp$ be that for $M_n^\perp$. Let $g = Ku \in R(K)$. Write u as $u = u_n + u_n^\perp$, where $u_n = P_n u$ and $u_n^\perp = P_n^\perp u$. Because both M_n and $M_n^\perp$ are invariant under K, $g = g_n + g_n^\perp$, where $g_n = P_n K u_n = K u_n$ and, similarly, $g_n^\perp = K u_n^\perp$. Consequently,

$$g - g_n = g_n^\perp = K(u - u_n) = K u_n^\perp = K_{n+1} u_n^\perp,\ \text{where } K_{n+1} = K|_{M_n^\perp}.$$

By (2.3) and the fact that $\lambda_{n+1} = \|K_{n+1}\|$, we have

$$\|g - g_n\| \leq \lambda_{n+1}\|u_n^\perp\| \leq \lambda_{n+1}\|u\|. \tag{2.4}$$

There are two possibilities. The first is that there are only a finite number n of nonzero eigenvalues, and $\lambda_{n+1} = 0$. This means $R(K) = M_n$ and so $g = g_n$. The second is that there are infinitely many nonzero eigenvalues. Since λ_n is a decreasing sequence bounded below, the limit $\lim_{n\to\infty} \lambda_n$ exists. Moreover, this limit is 0 because the only limit point of the nonzero eigenvalues is 0. Finally, this and $\|g - g_n\| \le |\lambda_{n+1}|\|u\|$ imply that

$$g = \lim_{n\to\infty} g_n = \sum_{k=1}^{\infty} \sum_{j=1}^{\dim \mathcal{E}_{\lambda_k}} \langle g, \phi_{k,j}\rangle \phi_{k,j},$$

from which the completeness of the basis for $R(K)$ follows immediately.

If we also have that $R(K)$ is dense in $\mathcal{H}$ – i.e., $\overline{R(K)} = \mathcal{H}$ –, then, since every vector in $R(K)$ can be expressed in terms of the basis, it follows from the theory in the notes on *Orthonormal Sets* that the set is an orthonormal basis for $\mathcal{H}$. □

We remark that (2.4) actually provides an estimate on the error made in approximating g by g_n.

Theorem 2.7 (Spectral Theorem)**.** *Let $K \neq 0 \in \mathcal{C}(\mathcal{H})$ be self-adjoint. Then, from among the eigenvectors of K, including those for $\lambda = 0$, we may select an orthonormal basis for $\mathcal{H}$.*

Proof. After proving the Fredholm alternative – Theorem 3.1 in the notes on *Several Important Theorems* –, we mentioned that the closure of the range of K satisfies $\overline{R(K)} = N(K^*)^{\perp}$. Since $K = K^*$ and $N(K)$ is closed, we have that $\mathcal{H} = \overline{R(K)} \oplus N(K)$. The basis constructed in Proposition 2.6 for $R(K)$ is also an orthonormal basis for $\overline{R(K)}$. (Why?) Since $N(K) = \mathcal{E}_{\lambda=0}$, we may contruct an orthornormal basis for it. Combining the two bases gives us an orthonormal basis for $\mathcal{H}$ composed of eigenvectors of K. □

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