X-ray Tomography & Integral Equations

by
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X-ray Tomography. An important part of X-ray tomography – the CAT scan – is solving a mathematical problem that goes back to the earlier twentieth century work of the mathematician Johann Radon: Suppose that there is a function[1] $f(x, y)$ defined in a region of the plane and that all we know about f is the collection of line integrals $\int_L f(x(s), y(s)ds$ over each line L that intersects the region. (See Figure 1.) The problem is to find f, given this information.

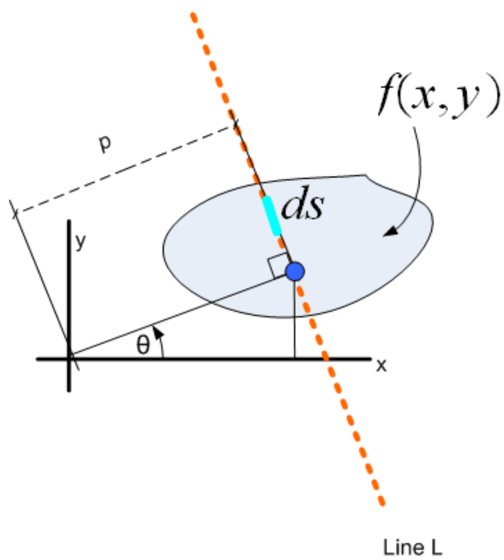

Figure 1: The region where f is defined and a typical line L cutting the region are shown. L is specified by ρ and the angle θ.

We will assume that the region where f is defined is a disk $D := \{|\mathbf{x}| \leq 1\}$. In Figure 1, the function is shown as having compact support in D. The unit vector $\mathbf{n}$ that is normal to L and points away from the origin is $\mathbf{n} = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}$. The tangent pointing upward is $\mathbf{t} = -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}$.

[1]This is an attenuation coefficient in a CAT scan.

If we let $s \geq 0$ be the arc length starting at the point $\rho\mathbf{n}$, then any point $\mathbf{x}$ above $\rho\mathbf{n}$ is specified by $\mathbf{x} = s\mathbf{t} + \rho\mathbf{n}$. If $\mathbf{x}$ is below $\rho\mathbf{n}$, then it is specified by $\mathbf{x} = -s\mathbf{t} + \rho\mathbf{n}$.

We will work with $\mathbf{x}$ above the vector $\rho\mathbf{n}$. Express $\mathbf{x}$ in terms of polar coordinates (r, ϕ), $\mathbf{x} = r\cos(\phi)\mathbf{i} + r\sin(\phi)\mathbf{j}$. Of course, $r = |\mathbf{x}|$. Comparing this with $\mathbf{x} = s\mathbf{t} + \rho\mathbf{n}$, we see that $r^2 = s^2 + \rho^2$ and $\rho = \mathbf{x}\cdot\mathbf{n} = r\cos(\phi - \theta)$. Since $\mathbf{x}$ is above $\rho\mathbf{n}$, we have that $\phi \geq \theta$ and thus $\phi = \theta + \mathrm{Cos}^{-1}(\rho/r)$. When $\mathbf{x}$ is below $\rho\mathbf{n}$, $\phi \leq \theta$ and $\phi = \theta - \mathrm{Cos}^{-1}(\rho/r)$. Breaking the integral $\int_L f(\mathbf{x}(s))ds$ into two pieces, making the change of variables $s = \sqrt{r^2 - \rho^2}$, $ds = (r^2 - \rho^2)^{-1/2} r dr$, and noting that $\rho \leq r \leq 1$, we have

$$\begin{aligned}\int_L f(\mathbf{x}(s))ds &= \int_{\phi\geq\theta} f(\mathbf{x}(s))ds + \int_{\theta\geq\phi} f(\mathbf{x}(s))ds \\ &= \int_\rho^1 \frac{f(r, \theta + \mathrm{Cos}^{-1}(\rho/r))rdr}{\sqrt{(r^2 - \rho^2}} + \int_\rho^1 \frac{f(r, \theta - \mathrm{Cos}^{-1}(\rho/r))rdr}{\sqrt{(r^2 - \rho^2}} \\ &= \int_\rho^1 \frac{\big(f(r, \theta + \mathrm{Cos}^{-1}(\rho/r)) + f(r, \theta - \mathrm{Cos}^{-1}(\rho/r))\big)rdr}{\sqrt{(r^2 - \rho^2}}.\end{aligned}$$

Assuming the $f\mathbf{x}) = f(r, \phi)$ is smooth enough, we can expand it in a Fourier series in ϕ,

$$f(r, \phi) = \sum_{n=-\infty}^{\infty} \widehat{f}_n(r) e^{in\phi}, \tag{1}$$

and then replace f in the integral on the right above by this series. Again making the assumption that interchanging sum and integral is possible and manipulating the resulting expression, we have

$$F(\rho, \theta) := \int_L f(\mathbf{x}(s))ds = 2\sum_{n=-\infty}^{\infty} e^{in\theta} \int_\rho^1 \widehat{f}_n(r) \frac{\cos(n\,\mathrm{Cos}^{-1}(\rho/r))rdr}{\sqrt{r^2 - \rho^2}}. \tag{2}$$

Since the line L is specified by the angle θ and distance ρ, the integral over L,is a function of θ and ρ, which we have denoted by $F(\rho, \theta)$. In addition, the expression $T_n(\rho/r) := \cos(n\mathrm{Cos}^{-1}(\rho/r))$ is actually an n^{th} degree Chebyshev polynomial. For example, $T_2(\rho/r) = 2\cos^2(\mathrm{Cos}^{-1}(\rho/r)) - 1 = 2(\rho/r)^2 - 1$. Using these two facts in connection with (2) we have

$$F(\rho, \theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} \int_\rho^1 2\widehat{f}_n(r) \frac{T_n(\rho/r)r}{\sqrt{r^2 - \rho^2}} dr, \tag{3}$$

which is the Fourier series for $F(\rho,\theta)$. It follows that the Fourier coefficients for $F(\rho,\theta)$ are given by

$$\widehat{F}_n(\rho) = \int_\rho^1 2\widehat{f}_n(r)\frac{T_n(\rho/r)r}{\sqrt{r^2-\rho^2}}dr,\ n \in \mathbb{Z}. \tag{4}$$

The point is that $F(\rho,\theta) = \int_L f(\mathbf{x}(s))ds$ is known, and so the Fourier coefficients $\widehat{F}_n(\rho)$ are all known. The problem of finding f, given F, is thus equivalent to solving the integral equations in (4) for the $\widehat{f}_n(r)$'s and recovering $f(r,\phi)$ from its Fourier series. In fact, these integral equations have exact solutions (see Keener, §3.7):

$$\widehat{f}_n(r) = -\frac{1}{\pi}\frac{d}{dr}\int_r^1 \frac{rT_n(\rho/r)\widehat{F}_n(\rho)}{\rho\sqrt{\rho^2-r^2}}d\rho,\ n \in \mathbb{Z}. \tag{5}$$

Classification of integral equations. Certain types of integral equations come up often enough that they are grouped into classes, which are described below. There, the function f and kernel $k(x,y)$ are known, u is the unknown function to be solved for, and λ is a parameter. The integral equations in (4) are Volterra equations of the first kind. Below is classification of the most common types of integral equations.

Fredholm Equations

1^{st} kind. $f(x) = \int_a^b k(x,y)u(y)dy$.
2^{nd} kind. $u(x) = f(x) + \lambda\int_a^b k(x,y)u(y)dy$.

Volterra Equations

1^{st} kind. $f(x) = \int_a^x k(x,y)u(y)dy$.
2^{nd} kind. $u(x) = f(x) + \lambda\int_a^x k(x,y)u(y)dy$.

Acknowledgments Figure 1 is from the article "A small note on Matlab iradon and the all-at-once vs. the one-at-a-time method," by Nasser M. Abbasi. July 17, 2008. The figure was downloaded on November 10, 2013, from the website

`http://12000.org/my_notes/note_on_radon/note_on_radon/note_on_radon.htm`

Previous: splines and finite elements
Next: bounded operators and closed subspaces