Test I

Instructions: *This test is due Monday, 27 March. You may consult any written or online source. You may* not *consult any person, either a fellow student or faculty member, except me.*

1. **(10 pts.)** Find the Hamiltonian description for the nonlinear mass-spring system,
$$m\ddot{x} + k_1 x + k_2 x^3 = 0.$$

2. **(10 pts.)** Using quadratic polynomials, approximate the first eigenvalue of $u'' + \lambda u = 0$, $u(0) = 0$, $u(1) + u'(1) = 0$.

3. Geodesics on the unit sphere $\mathbb{S}^2$ are extremals for the functional
$$J[\mathbf{x}(t)] = \int_a^b \sqrt{\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2}\,dt,$$
where $\mathbf{x}$ is a point on the unit sphere and (θ, ϕ) are its spherical coordinates; $\mathbf{x}(t)$ is a curve parametrized by $t \in [a, b]$.
   (a) **(10 pts.)** Take the starting point as the north pole of the sphere and assume that $t = \theta$, so the we are working with $\phi = \phi(\theta)$. Show that the extremals are just the great circles given by $\phi =$ constant.
   (b) **(5 pts.)** Show that on any of these extremals the only point conjugate to the north pole is the south pole.

4. **(10 pts.)** Let $F(z)$ be entire (i.e., analytic on $\mathbb{C}$). If F is bounded on the real axis and satisfies the overall bound $|F(z)| \le Ae^{\sigma|z|}$, then $|F(z)| \le Be^{\sigma|\mathrm{Im}(z)|}$. (Hint: $G(z) = F(z)e^{i(\sigma+\epsilon)z}$ satisfies $|G(iy)| \le A$ and $|F(x)| \le \sup_{x\in\mathbb{R}} |G(x)|$. Apply the Phragmén-Lindelöf theorem for sectors that you derived in assignment 3.)

5. **(15 pts.)** Show that the map $z = \frac{1}{2}(w + 1/w)$ takes a circle $|w| = a > 1$ onto an ellipse in the z-plane, and that the exterior of the circle, $|w| > a$ corresponds to the exterior of the ellipse. Next, show that $\zeta = i\frac{w-a}{w+a}$ maps $|w| = a$ to $\mathrm{Im}(\zeta) = 0$, and $|w| > a$ onto the upper half plane $\mathrm{Im}(\zeta) > 0$. Use the resulting map to produce a flow that has constant velocity at $z = \infty$ and has the ellipse found earlier as a streamline.

6. **(10 pts.)** The Hermite polynomials $H_n(x)$ satisfy the recurrence relation,

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \;\; H_0(x) = 1 \text{ and } H_1(x) = 2x.$$

Use this to show that the generating function for the Hermite polynomials is

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{2tx - t^2}.$$

7. Consider the operator $Lu = -u''$ defined on functions in $L^2[0, \infty)$ having u'' in $L^2[0, \infty)$ and satisfying the boundary condition that $u'(0) = 0$; that is, L has the domain

$$\mathcal{D}_L = \{u \in L^2[0, \infty) \,|\, u'' \in L^2[0, \infty) \text{ and } u'(0) = 0\}.$$

(a) **(5 pts.)** Show that at least formally L is self-adjoint.

(b) **(10 pts.)** Find the Green's function $G(x, \xi; z)$ for $-G'' - zG = \delta(x - \xi)$, with $G'(0, \xi; z) = 0$. (This is the kernel for the resolvent $(L - zI)^{-1}$.)

(c) **(15 pts.)** Employ the spectral theorem, as we did in class, to obtain the cosine transform formulas,

$$F(\mu) = \int_0^{\infty} f(x) \cos(\mu x) dx \text{ and } f(x) = \frac{2}{\pi} \int_0^{\infty} F(\mu) \cos(\mu x) d\mu\,.$$