Final Examination

This take-home exam is due at 3 pm on Tuesday, May 8. You may consult any written or online source. You may *not* consult any person, either a fellow student or faculty member, except your instructor

1. **(5 pts.)** Let T be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. In addition, suppose that 0 is in the continuous spectrum of T. If D is the range of T, show that $H = T^{-1}$, with domain H being D, is a self-adjoint operator.

2. Let $Lu = -r^{-2}\frac{d}{dr}r^2\frac{du}{dr}$, $0 < r < \infty$, with boundary conditions $r^2u' \to 0$ and u bounded as $r \to 0$, and similar conditions as $r \to +\infty$.

   (a) **(5 pts.)** Show that L is (formally) self adjoint in the inner product $\langle u, v\rangle = \int_0^\infty u(r)\overline{v(r)}r^2 dr$.

   (b) **(10 pts.)** Find the Green's function for L.

   (c) **(15 pts.)** Use Stone's formula (or the book's contour technique) to find the associated spectral transform. In addition, you may make use of any asymptotic formulas required in the problem.

3. **(15 pts.)** Problem 6(a), section 7.5, p. 333 in the text.

4. **(10 pts.)** Consider $E_1(x) = \int_x^\infty t^{-1}e^{-t}dt$, $x > 0$. Find the asymptotic expansion for $E_1(x)$ as $x \to +\infty$. (Hint: show that $E_1(x) = e^{-x}\int_0^\infty (1+t)^{-1}e^{-xt}dt$.)

5. **(10 pts.)** Problem 11(a), section 10.3, p. 465 in the text.

6. **(15 pts.)** Consider $f(x) = \int_0^2 e^{ix(t^2-2t)}dt$, $x > 0$. Use the method of steepest descent to show that $f(x) = e^{i(\pi/4 - x)}\sqrt{\frac{\pi}{x}}(1 + \mathcal{O}(x^{-1/2}))$ as $x \to +\infty$.

7. **(15 pts.)** Prove this version of the principle of stationary phase: For all $\lambda \in \mathbb{R}$, let $F(\lambda) := \int_{-\infty}^\infty e^{i\lambda h(t)}g(t)dt$, where $g \in C^{(2)}(\mathbb{R})$, $g, g' \in L^1(\mathbb{R})$, $g(0) \neq 0$, and where $h \in C^{(3)}(\mathbb{R})$ is real valued, and satisfies $h'(0) = 0$, $h''(t) > 0$ for all $t \in \mathbb{R}$. Then, $F(\lambda) \sim \sqrt{\frac{2\pi}{\lambda h''(0)}}g(0)e^{i\lambda h(0) + i\pi/4}$ as $\lambda \to +\infty$.