Final Examination

This take-home exam is due at 4 pm on Wednesday, May 7. You may consult any written or online source. You may *not* consult any person, either a fellow student or faculty member, except your instructor

1. **(15 pts.)** Suppose that L is a closed, densely defined self-adjoint linear operator on a Hilbert space $\mathcal{H}$, with domain $D(L)$. Show that the spectrum of L is a subset of $\mathbb{R}$ and that the residual spectrum of L is empty.

2. Consider the operator $Lu = -u''$ defined on functions in $L^2[0,\infty)$ having u'' in $L^2[0,\infty)$ and satisfying the boundary condition that $u'(0) = 0$; that is, L has the domain

$$\mathcal{D}_L = \{u \in L^2[0,\infty) \,|\, u'' \in L^2[0,\infty) \text{ and } u'(0) = 0\}.$$

   (a) **(10 pts.)** Find the Green's function $G(x,\xi;z)$ for $-G'' - zG = \delta(x-\xi)$, with $G_x(0,\xi;z) = 0$. (This is the kernel for the resolvent $(L - zI)^{-1}$.)

   (b) **(10 pts.)** Employ the spectral theorem to obtain the cosine transform formulas,

$$F(\mu) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\mu x)dx \text{ and } f(x) = \int_0^\infty F(\mu)\cos(\mu x)d\mu\,.$$

3. **(15 pts.)** Use the following convention to define the Fourier transform: $\mathcal{F}[f](\xi) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\xi x}dx$, so $\|f\| = \|\hat{f}\|$ and $\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-i\xi x}d\xi$. You are given that the eigenvalue problem $-y_n'' + x^2 y_n = (2n+1)y_n$, where $n = 0, 1, 2\ldots$, $y \in L^2(\mathbb{R})$ has a unique solution that is even or odd, depending on whether n is even or odd. Show that $\mathcal{F}[y_n] = (-1)^n y_n$.

4. **(15 pts.)** Suppose that $g \in C^\infty(\mathbb{R})$ satisfies

$$|g^{(m)}(t)| \le c_m(1+t^2)^{n_m}$$

   for all nonnegative integers m. Here c_m and n_m depend on g and m. Show that if f is in Schwartz space, $\mathcal{S}$, then $fg \in \mathcal{S}$. In addition, suppose $T \in \mathcal{S}'$, show that $g(x)T(x)$ is also in $\mathcal{S}'$, if $\langle gT, f\rangle := \langle T, gf\rangle$.

5. **(20 pts.)** Prove this version of Watson's Lemma: Suppose that $z \in \mathbb{C}$ and that $|\arg(z)| \le \delta < \frac{pi}{2}$. Let $F(z) := \int_{-\infty}^{\infty} e^{-zt^2} f(t)dt$, where for $t \in \mathbb{C}$, $|t| \le T_0$, $f(t) = \sum_{n=0}^{\infty} a_n t^n$ and, in addition, there is an $\alpha > 0$ such that $|f(t)| \le C|t|^\alpha$, $|t| \ge T_0$. Then,

$$F(z) \sim \sum_{k=0}^{\infty} a_{2k} \Gamma\big(k + \frac{1}{2}\big) z^{-k-\frac{1}{2}}, \ |z| \to \infty.$$

6. The object of this problem is to prove Stirling's formula for $\Gamma(x+1)$, $x \to +\infty$.

   (a) **(5 pts.)** Show that $x^{-x-1} e^x \Gamma(x+1) = \int_0^\infty e^{-xh(t)} dt$, $h(t) := t - 1 - \log(t)$.

   (b) **(5 pts.)** Let $u = u(t) := \sqrt{\frac{h(t)}{(t-1)^2}}(t-1)$. Verify that $u(t) \in C^1(0,\infty)$, is increasing, and that

   $$x^{-x-1} e^x \Gamma(x+1) = \int_{-\infty}^{\infty} e^{-xu^2} \frac{dt}{du} du.$$

   (c) **(5 pts.)** Show that for u near 0, $dt/du = \sqrt{2} + \mathcal{O}(u)$. Use the previous problem to show that

   $$\Gamma(x+1) \sim \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} (1 + \mathcal{O}(x^{-1})).$$