Residues and Contour Integration Problems

Classify the singularity of $f(z)$ at the indicated point.

1. $f(z) = \cot(z)$ at $z = 0$. Ans. Simple pole.

   Solution. The test for a simple pole at $z = 0$ is that $\lim_{z\to 0} z\cot(z)$ exists and is not 0. We can use L' Hôpital's rule:

   $$\lim_{z\to 0} z\cot(z) = \lim_{z\to 0} \frac{z\cos(z)}{\sin(z)} = \lim_{z\to 0} \frac{\cos(z) - z\sin(z)}{\cos(z)} = 1.$$

   Thus the singularity is a simple pole.

2. $f(z) = \frac{1+\cos(z)}{(z-\pi)^2}$ at $z = \pi$. Ans. Removable.

   Solution. Power series is the simplest way to do this. We can expand $\cos(z)$ in a Taylor series about $z = \pi$. To do so, use the trig identity $\cos(z) = -\cos(z - \pi)$. Next, expand $1 - \cos(z - \pi)$ in a power series in $z - \pi$:

   $$1 + \cos(z) = 1 - \cos(z - \pi) = \frac{1}{2!}(z - \pi)^2 - \frac{1}{4!}(z - \pi)^4 + \cdots$$

   From this, we get

   $$\frac{1 + \cos(z)}{(z - \pi)^2} = \frac{(z - \pi)^2(\frac{1}{2} - \frac{1}{4!}(z - \pi)^2 + \cdots)}{(z - \pi)^2} = \frac{1}{2} - \frac{1}{4!}(z - \pi)^2 + \cdots ,$$

   which is the Laurent series for $\frac{1+\cos(z)}{(z-\pi)^2}$. Since there are no negative powers in the series, the singularity is removable.

3. $f(z) = \sin(1/z)$. Ans. Essential singularity.

4. $f(z) = \frac{z^2-z}{z^2+2z+1}$ at $z = -1$. Ans. Pole of order 2.

5. $f(z) = z^{-3}\sin(z)$ at $z = 0$. Ans. Pole of order 2.

6. $f(z) = \csc(z)\cot(z)$ at $z = 0$. Ans. Pole of order 2.

Find the residue of $g(z)$ at the indicated singulatity.

7. $g(z) = \frac{1}{z^2+1}$ at $z = -i$. Ans. $\mathrm{Res}_{-i}(g) = \frac{1}{2}i$.

   Solution. Since $g(z) = \frac{1}{(z-i)(z+i)}$, we have that $(z+i)g(z) = \frac{1}{z-i}$, which is analytic and nonzero at $z = -i$. Hence, $g(z)$ has a simple pole at $z = -i$. The residue is thus $\mathrm{Res}_{-i}(g) = \lim_{z\to -i}(z+i)g(z) = \frac{1}{-2i} = \frac{1}{2}i$

8. $g(z) = \frac{e^z}{z^3}$ at $z = 0$. Ans. $\mathrm{Res}_0(g) = \frac{1}{2}$.

   Solution. Using the power series for e^z, we see that the Laurent series for $g(z)$ about $z = 0$ is

   $$\frac{e^z}{z^3} = \frac{1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \cdots}{z^3} = z^{-3} + z^{-2} + \frac{1}{2!}z^{-1} + \frac{1}{3!} + \frac{1}{4!}z + \cdots$$

   The the residue is a_{-1}, the coefficient of z^{-1}. Hence, $\mathrm{Res}_0(g) = a_{-1} = \frac{1}{2}$.

9. $g(z) = \tan(z)$ at $z = \pi/2$. Ans. $\mathrm{Res}_{\pi/2}(g) = -1$.

10. $g(z) = \frac{z+2}{(z^2-2z+1)^2}$ at $z = 1$. Ans. $\mathrm{Res}_1(g) = 1$.

11. $g(z) = f(z)/h(z)$ at $z = z_0$, given that $f(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$. Show that $z = z_0$ is a simple pole and find $\mathrm{Res}_{z_0}(g)$. Ans. $\mathrm{Res}_{z_0}(g) = f(z_0)/h'(z_0)$.

The singularities for the functions below are all simple poles. Find all of them and use exercise 11 above to find the residues at them.

12. $g(z) = \frac{z^2-1}{z^2-5iz-4}$. Ans. The singularities are at i and $4i$ and the residues are $\mathrm{Res}_i(g) = -\frac{2}{3}i$ and $\mathrm{Res}_{4i}(g) = \frac{17}{3}i$.

    Solution. The singularities are the roots of $z^2 - 5iz - 4 = 0$, which are i and $4i$. In our case, the functions f and h in exercise 11 are $f(z) = z^2-1$ and $h(z) = z^2-5iz-4$, and $f(z)/h'(z) = (z^2-1)/(2z-5i)$. It immediately follows that

    $$\mathrm{Res}_i(g) = \frac{i^2-1}{2i-5i} = \frac{-2}{-3i} = -\frac{2}{3}i.$$

    The other residue follows similarly.

13. $g(z) = \tan(z)$. Ans. The singularities are at $z_n = (n + \frac{1}{2})\pi$, where $n = 0, \pm 1, \pm 2, \ldots$, and the residues at z_n are $\mathrm{Res}_{z_n}(g) = -1$.

14. $g(z) = \frac{z^2}{z^3 - 8}$. Ans. The singularities are at the roots of $z^3 - 8 = 0$. There are three of these: 2, $2e^{2i\pi/3}$ and $2e^{4i\pi/3}$. The residues at these three points are all $1/3$.

15. $g(z) = \frac{e^z}{\sin(z)}$. Ans. The singularities are at the roots of $\sin(z) = 0$, which are $n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, and the residues there are $\mathrm{Res}_{n\pi}(g) = (-1)^n e^{n\pi}$.

16. $g(z) = \frac{\sin(z)}{z^2 - 3z + 2}$. Ans. The singularities are at the roots of $z^2 - 3z + 2 = 0$, which are 1 and 2. The residues are $\mathrm{Res}_1(g) = -\sin(1)$ and $\mathrm{Res}_2(g) = \sin(2)$.

Use the residue theorem to evaluate the contour intergals below. Where possible, you may use the results from any of the previous exercises.

17. $\oint_C \frac{z^2}{z^3 - 8} dz$, where C is the counterclockwise oriented circle with radius 1 and center $3/2$. Ans. $2\pi i/3$.

    Solution. From exercise 14, $g(z)$ has three singularities, located at 2, $2e^{2i\pi/3}$ and $2e^{4i\pi/3}$. A simple sketch of C shows that only 2 is inside of C. Thus, by the residue theorem and exercise 14, we have

$$\oint_C \frac{z^2}{z^3 - 8} dz = 2\pi i \,\mathrm{Res}_2(g) = 2\pi i/3 = 2\pi i/3.$$

18. $\oint_C \frac{z^2}{z^3 - 8} dz$, where C is the counterclockwise oriented circle with radius 3 and center 0. Ans. $2\pi i$.

19. $\oint_C \frac{z^2 - 1}{z^2 - 5iz - 4} dz$, where C is any simple closed curve that is positively oriented (i.e., counterclockwise) and encloses the following points: (a) only i; (b) only $4i$; (c) both i and $4i$; (d) neither i nor $4i$. Ans. (a) $4\pi/3$. (b) $-34\pi/3$. (c) -10π. (d) 0.

20. $\oint_C \frac{e^z}{\sin(z)} dz$, where C is the positively traversed rectangle with corners $-\pi/2 - i$, $5\pi/2 - i$, $-\pi/2 + 2i$ and $5\pi/2 + 2i$. Ans. $2\pi i(1 - e^{\pi} + e^{2\pi})$.

21. $\oint_C \frac{z+2}{(z^2-2z+1)^2}dz$, where C is the positively oriented semicircle that is located in the *right half plane* and has center 0, radius $R > 1$, and diameter located on the imaginary axis. Ans. 0.

Solution. From exercise 10, the only singularity of the integrand is at 1. By the residue theorem and exercise 10, we have

$$\oint_C \frac{z+2}{(z^2-2z+1)^2}dz = 2\pi i \operatorname{Res}_1(g) = 2\pi i \cdot 1 = 2\pi i.$$

22. $\oint_C \frac{1}{(z^2+1)(z^2+4)}dz$, where C is the *negatively* oriented (i.e., clockwise) semicircle that is located in the *upper half plane* and has center 0, radius $R > 2$, and diameter located on the real axis. Ans. $-\pi/6$.

Find the values of the definite integrals below by contour-integral methods.

23. $\int_0^{2\pi} \frac{d\theta}{5-3\sin(\theta)}$. Ans. $\pi/2$.

Solution. Begin by converting this integral into a contour integral over C, which is a circle of radius 1 and center 0, oriented positively. To do this, let $z = e^{i\theta}$. Note that $dz = ie^{i\theta}d\theta = izd\theta$, so $d\theta = dz/(iz)$. Also, $\sin(\theta) = (z - z^{-1})/(2i)$. We thus have

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\sin(\theta)} = \oint_C \frac{dz}{iz(5 - \frac{3z-3/z}{2i})} = \oint_C \frac{(-2)dz}{3z^2 - 10iz - 3}.$$

The integrand has singularities at $z_\pm = (10i \pm 8i)/6 = \begin{cases} 3i \\ i/3. \end{cases}$ Only $z_- = i/3$ is inside C. It is a simple pole because the integrand has the form $f(z)/(z - i/3)$, where f is analytic at $i/3$. Using exercise 11, we see that

$$\operatorname{Res}_{i/3}\left(\frac{(-2)}{3z^2 - 10iz - 3}\right) = \frac{-2}{6z_- - 10i} = \frac{-2}{6i/3 - 10i} = -i/4.$$

The residue theorem then implies that

$$\oint_C \frac{(-2)dz}{3z^2 - 10iz - 3} = 2\pi i \operatorname{Res}_{i/3}\left(\frac{(-2}{3z^2 - 10iz - 3}\right) = \pi/2.$$

24. $\int_0^{2\pi} \frac{d\theta}{3-2\cos(\theta)}$. Ans. $2\pi/\sqrt{5}$.

Solution. Begin by converting this integral into a contour integral over C, which is a circle of radius 1 and center 0, oriented positively. To do this, let $z = e^{i\theta}$. Note that $dz = ie^{i\theta}d\theta = izd\theta$, so $d\theta = dz/(iz)$. Also, $\cos(\theta) = (z + z^{-1})/2$. We thus have

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos(\theta)} = \oint_C \frac{dz}{iz(3 - z - 1/z)} = \oint_C \frac{idz}{z^2 - 3z + 1}.$$

The integrand has singularities at $z_\pm = (3 \pm \sqrt{5})/2$. Only $z_- = (3 - \sqrt{5})/2$ is inside C. It is a simple pole because the integrand has the form $f(z)/(z - (3 - \sqrt{5})/2))$, where f is analytic at $(3 - \sqrt{5})/2$. Using exercise 11, we see that

$$\mathrm{Res}_{3-\sqrt{5})/2}\left(\frac{i}{z^2 - 3z + 1}\right) = \frac{i}{2z_{-}3} = -\frac{i}{\sqrt{5}}.$$

The residue theorem then implies that

$$\oint_C \frac{idz}{z^2 - 3z + 1} = 2\pi i \,\mathrm{Res}_{3-\sqrt{5})/2}\left(\frac{i}{z^2 - 3z + 1}\right) = \frac{2\pi}{\sqrt{5}}.$$

25. $\int_0^{2\pi} \frac{d\theta}{5-4\sin(\theta)}$. Ans. $2\pi/3$.

26. $\int_0^{2\pi} \frac{\cos(\theta)d\theta}{13+12\cos(\theta)}$. Ans. $-4\pi/15$. (This has two simple poles within the contour.)

27. $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}dx$. Ans. $\pi/6$. (Hint: reverse the contour in exercise 22 and let $R \to \infty$.)