Linear Filters

§**1. Convolutions and filters.** A filter is a "black box" that takes an input signal, processes it, and then returns an output signal that in some way modifies the input. For example, if the input signal is noisy, then one would want a filter that removes noise, but otherwise leaves the signal unchanged.

From a mathematical point of view, a signal is a function $f : \mathbb{R} \to \mathbb{C}$ that is piecewise contiuous, and perhaps satisfies other properties, and a filter is a transformation L that maps a signal f into another signal $\tilde{f}$. The space of signals is a vector space. L is a *linear filter* if it is a linear transformation on the space of signals—*i.e.,* it satisfies these two properties:

(i) Additivity: $L[f+g] = L[f] + L[g]$

(ii) Homogeneity: $L[cf] = cL[f]$, where c is a constant.

There is another property that we want our "black box" L to have. If we play an old, scratchy record for half an hour starting at 3 pm today, and put the signal through a noise reducing filter, we will hear the cleaned-up output, at roughly the same time as we play the record. If we play the same record at 10 am tomorrow morning, and use the same filter, we should hear the identical output, again at roughly the same time. This property is called *time invariance.* It can be precisely formulated this way. Let $f(t)$ be the original signal, and let $L[f(t)] = \tilde{f}(t)$ be the filtered output. L is said to be *time-invariant* if $L[f(t-a)] = \tilde{f}(t-a)$ for any a. In words, L is time invariant if the time-shifted input signal $f(t-a)$ is transformed by L into the time-shifted output signal $\tilde{f}(t-a)$. Not every linear transformation has this property. For example, it fails for the linear transformation $f \mapsto \int_0^t f(\tau)d\tau$.

A linear time-invariant transformation or filter L is closely related to a convolution product. To see how the two are related, we will determine what function $e^{i\omega t}$ is mapped to by L.

Proposition 1.1: *Let L be a linear, time-invariant transformation on the space of signals that are piecewise continuous functions, and let ω be a real number. Then,*

$$L[e^{i\omega t}] = \sqrt{2\pi}\,\hat{h}(\omega)e^{i\omega t}.$$

Proof: Let $L[e^{i\omega t}] = \tilde{e}_\omega(t)$. Because L is time-invariant, we have that

$$L[e^{i\omega(t-a)}] = \tilde{e}_\omega(t-a).$$

Since $e^{i\omega(t-a)} = e^{i\omega t}e^{-i\omega a}$ and since L is linear, we also have that

$$\begin{aligned} L[e^{i\omega(t-a)}] &= L[e^{i\omega t}e^{-i\omega a}] \\ &= e^{-i\omega a}L[e^{i\omega t}] \\ &= e^{-i\omega a}\tilde{e}_\omega(t) \end{aligned}$$

Comparing the two previous equations, we find that

$$\tilde{e}_\omega(t-a) = e^{-i\omega a}\tilde{e}_\omega(t).$$

Now, a is arbitrary. We may set $a = t$ in the previous equation. Solving for $\tilde{e}(t)$ yields

$$\tilde{e}_\omega(t) = \tilde{e}_\omega(0)e^{i\omega t}.$$

Letting $\hat{h}(\omega) = \tilde{e}_\omega(0)/\sqrt{2\pi}$ completes the proof. ∎

The function $\hat{h}(\omega)$, which should be thought of as the Fourier transform of a function h, completely determines L. To see this, we first apply L to a signal f, but with f written in a peculiar way,

$$f(t) = \mathcal{F}^{-1}[\hat{f}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} d\omega.$$

Using this form for f, we get the following chain of equations:

$$\begin{aligned}
L[f] &= L\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} d\omega\right] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)L[e^{i\omega t}]d\omega \quad \text{[linearity]} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\left(\sqrt{2\pi}\hat{h}(\omega)\right)e^{i\omega t} d\omega \quad \text{[Proposition1.1]} \\
&= \sqrt{2\pi}\mathcal{F}^{-1}[\hat{f}(\omega)\hat{h}(\omega)] \quad \text{[def'n of inverse FT]} \\
&= f * h(t) \quad \text{[Convolution Theorem]}
\end{aligned}$$

The argument presented above is only "formal," because the interchange of L and the integral in the second step has not been justified. Nevertheless, the result that we arrived at is true with very few restrictions on either L or the space of signals being considered.

There are physical interpretations for both $h(t)$ and $\hat{h}(\omega)$. Assume that $h(t)$ is continuous, that δ is a small positive number. We want to apply L to an impulse: a signal $f_\delta(t)$ that is any piecewise continuous function that is 0 outside of an interval $[-\delta, \delta]$ and satisfies $\int_{-\infty}^{\infty} f_\delta(t)dt = 1$. Applying L to f_δ we get

$$\begin{aligned}
L[f_\delta(t)] &= f_\delta * h(t) \\
&= \int_{-\infty}^{\infty} f_\delta(\tau)h(t-\tau)d\tau \\
&= \int_{-\delta}^{\delta} f_\delta(\tau)h(t-\tau)d\tau \\
&\approx h(t)\underbrace{\int_{-\delta}^{\delta} f_\delta(\tau)d\tau}_{1} = h(t)
\end{aligned}$$

Thus $h(t)$ is the approximate response to an impulse signal. For that reason $h(t)$ is called the response function. We have already seen that $L[e^{i\omega t}] = \sqrt{2\pi}\hat{h}(\omega)e^{i\omega t}$. Thus $\hat{h}(\omega)$ is the amplitude to of the response to a "pure frequency" signal; it is called the *system function*.

§**2. Causality and the design of filters.** We want to design a device that will cut off a signal's frequencies if they are beyond a certain range. Usually, such filters are employed in noise reduction, and are called *low-pass* filters.

A simple guess at how to do this is to first look at the effect of a filter in the "frequency domain." Since $L[f] = f * h$, we have, by the Convolution Theorem,

$$\widehat{L[f]}(\omega) = \sqrt{2\pi}\hat{f}(\omega)\hat{h}(\omega).$$

To remove all frequencies beyond a certain range, and still leave the other frequency components untouched, we need only choose

$$\hat{h}_{\omega_c}(\omega) := \begin{cases} 1/\sqrt{2\pi} & \text{if } |\omega| \le \omega_c \\ 0 & \text{if } |\omega| > \omega_c \end{cases},$$

where ω_c is some cut-off frequency that is at our disposal. The response function corresponding to the system function $\hat{h}_{\omega_c}$ is easy to calculate:

$$\begin{aligned} h_{\omega_c}(t) &= \mathcal{F}^{-1}[\hat{h}_{\omega_c}] \\ &= (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} \hat{h}_{\omega_c}(\omega) e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{i\omega t} d\omega \\ &= \left[\frac{e^{i\omega t}}{2it\pi} \right]_{\omega=-\omega_c}^{\omega=\omega_c} \\ &= \frac{e^{i\omega_c t} - e^{-i\omega_c t}}{2it\pi} \\ &= \frac{\sin(\omega_c t)}{\pi t} \end{aligned}$$

We want to see whether h_{ω_c} is a reasonable response function, and so we will see what happens when we filter a very simple input function,

$$f_{t_c}(t) =:= \begin{cases} 1 & \text{if } 0 \le t \le t_c \\ 0 & \text{if } t < 0 \text{ or } t > t_c \end{cases},$$

where t_c is the switch-off time. The effect of the filter described above is

$$\begin{aligned} L_{\omega_c}[f_{t_c}](t) &= \int_{-\infty}^{\infty} f_{t_c}(\tau) h_{\omega_c}(t-\tau) d\tau \\ &= \int_0^{t_c} \frac{\sin(\omega_c(t-\tau))}{\pi(t-\tau)} d\tau \\ &= \frac{1}{\pi} \int_{\omega_c(t-t_c)}^{\omega_c t} \frac{\sin u}{u} du \\ &= \frac{1}{\pi} \left\{ \text{Si}(\omega_c t) - \text{Si}(\omega_c(t-t_c)) \right\}, \end{aligned}$$

where $\mathrm{Si}(z) = \int_0^z \frac{\sin u}{u} du$.

We want to plot $L_{\omega_c}[f_{t_c}](t) = \frac{1}{\pi}\{\mathrm{Si}(\omega_c t) - \mathrm{Si}(\omega_c(t - t_c))\}$. We need to pick numerical values for t_c and ω_c; let us choose both parameters to be 1. The result is shown below.

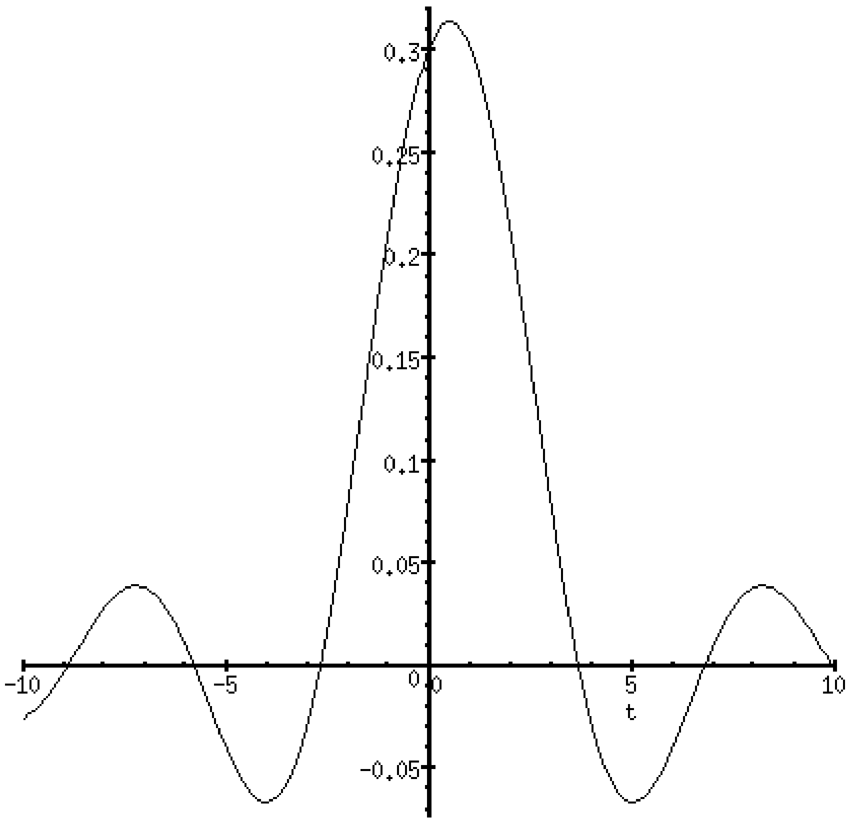

Figure 1: A plot of $\frac{1}{\pi}\{\mathbf{Si}(t) - \mathbf{Si}(t-1)\}$

Note that an appreciable portion of the graph occurs before $t = 0$. In physical terms, this means that the output signal is occurring before the input signal has arrived! Since this violates causality, the filter that we have constructed cannot be physically realized. Causality must be taken into account when designing filters.

§**3. Causal filters.** A causal filter is one for which the output signal begins *after* the input signal has started to arrive. The result below tells us which filters are causal.

PROPOSITION 3.1: *Let h be a piecewise continuous function for which the integral $\int_{-\infty}^{\infty} |h(t)|dt$ is finite. Then, h is the impulse response function for a causal filter if and only if $h(t) = 0$ for all $t < 0$.*

PROOF: We will prove that if $h(t) = 0$ for all $t < 0$, then the corresponding filter is causal. We leave the converse as an exercise. Suppose that a signal starts arriving at $t = 0$—that is, $f(t) = 0$ for all $t < 0$. The output of the filter is then

$$L[f](t) = f * h(t) = \int_0^{\infty} f(\tau)h(t - \tau)d\tau,$$

where the lower limit in the integral above is 0 because $f(\tau) = 0$ when $\tau < 0$. Now, if a negative value of t is inserted in the formula above, the argmuent of h is $t - \tau$, which satisfies $t - \tau \le t < 0$, so $h(t-\tau) = 0$ for all $\tau \ge 0$. The integral thus vanishes, and $f * h(t) = 0$ for all such t. The time-invariance of the filter takes care of the case when a signal starts arriving at a time other than $t = 0$. ∎

This proposition applies to the response function, but it also gives us important information about the system function, $\hat{h}(\omega)$. Recall that the system function is the Fourier transform of h. If the filter is causal, then this implies

$$\begin{aligned}\hat{h}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} h(t)e^{-i\omega t}dt \quad \text{[causality]} \\ &= \mathcal{L}[h(t)/\sqrt{2\pi}](i\omega) \quad [\mathcal{L} = \text{Laplace transform}]\end{aligned}$$

In words, the system function is the Laplace transform of a multiple of the response function, with the Laplace transform variable $s = i\omega$. The converse is also true.

PROPOSITION 3.2: *Suppose that $G(s) = \mathcal{L}[g](s)$ and that $\int_0^\infty |g(t)|dt$ is convergent. Then, the function $G(i\omega)/\sqrt{2\pi}$ is the system function for a causal filter L, and the function*

$$h(t) := \begin{cases} g(t) & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases}$$

is the response function for L.

PROOF: If we start with the h defined in the proposition, and take its Fourier transform, then we have

$$\begin{aligned}\hat{h}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} g(t)e^{-i\omega t}dt \\ &= \frac{1}{\sqrt{2\pi}}\mathcal{L}[g(t)](i\omega) \\ &= G(i\omega)/\sqrt{2\pi} \quad ∎\end{aligned}$$

One of the older causal, noise reducing filters is the *Butterworth*. It is consructed using Proposition 3.2 with $g(t) = Ae^{-\alpha t}$, where $\alpha > 0$ and $A > 0$ are parameters. Since $G(s) = \mathcal{L}[Ae^{-\alpha t}](s) = A(\alpha + s)^{-1}$, we have

$$\text{Butterworth filter:} \begin{cases} h(t) = \begin{cases} Ae^{-\alpha t} & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases} \\ \hat{h}(\omega) = (A/\sqrt{2\pi})(\alpha + i\omega)^{-1} \end{cases}$$

Reference: A. Papoulis, *The Fourier Integral and its Applications*, McGraw-Hill, New York, 1962.