# Properties of the Fourier Transform

**Definition:** $\mathcal{F}[f](\omega) := \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx.$

- Notation: $\hat{f}(\omega) := \mathcal{F}[f](\omega)$.
- Linearity: $\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$, where $a, b$ are constants.
- Continuity of the FT: *If* $\int_{-\infty}^{\infty} |f(x)|dx < \infty$, *then* $\hat{f}$ *is continuous in* $\omega$.
- Important formulas & facts:
  1. Multiplication by $x^n$: $\mathcal{F}[x^n f] = i^n \hat{f}^{(n)}$.
  2. Derivatives: $\mathcal{F}[f^{(n)}] = (i\omega)^n \hat{f}$.
  3. Translations: $\mathcal{F}[f(x-a)] = e^{-i\omega a}\hat{f}(\omega)$.
  4. Complex conjugates: $\overline{\mathcal{F}[f](\omega)} = \mathcal{F}[\,\overline{f}\,](-\omega)$.
  5. Gaussians: $\mathcal{F}[e^{-\alpha x^2}] = \sqrt{\frac{\pi}{\alpha}}e^{-\omega^2/(4\alpha)}$

**Inversion formula:** $\dfrac{f(x^+) + f(x^-)}{2} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega.$

- Notation: $\mathcal{F}^{-1}[g](x) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{i\omega x}d\omega.$
- Linearity: $\mathcal{F}^{-1}[af + bg] = a\mathcal{F}^{-1}[f] + b\mathcal{F}^{-1}[g]$, where $a, b$ are constants.
- Important formulas & facts:
  1. Derivatives: $\mathcal{F}^{-1}[\hat{f}^{(n)}] = (-ix)^n f$
  2. Multiplication by $\omega^n$: $\mathcal{F}^{-1}[\omega^n \hat{f}] = (-i)^n f^{(n)}$.
  3. Multiplication by $e^{-i\omega a}$: $\mathcal{F}^{-1}[e^{-i\omega a}\hat{f}](x) = f(x-a)$.
  4. $\mathcal{F}^{-1}[g](z) = \dfrac{1}{2\pi}\mathcal{F}[g](-z)$
  5. Gaussians: $\mathcal{F}^{-1}[e^{-\alpha\omega^2}] = \frac{1}{\sqrt{4\pi\alpha}}e^{-x^2/(4\alpha)}$

**Convolution product:** $H * K(u) := \int_{-\infty}^{\infty} H(v)K(u-v)dv.$

**Convolution Theorem:**

- $\mathcal{F}[f * g] = \hat{f}\hat{g}$ and $\mathcal{F}^{-1}[\hat{f}\hat{g}] = f * g$.
- $\mathcal{F}^{-1}[\hat{f} * \hat{g}] = 2\pi fg$ and $\mathcal{F}[fg] = \dfrac{1}{2\pi}\hat{f} * \hat{g}$.

**Parseval's Formula:** $\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{g}(\omega)}d\omega.$