Quiz 3 – Key

Instructions: Show all work in the space provided. No notes, calculators, cell phones, etc. are allowed.

1. Define the terms below.

   (a) **(5 pts.)** $\lim_{x\to a} f(x)$ does *not* exist – Let $L \in \mathbb{R}$. We will first define $\lim_{x\to a} f(x) \neq L$: For some $\varepsilon_0 > 0$ and every $\delta > 0$ there is an x satisfying $0 < |x-a| < \delta$ for which $|f(x) - L| \geq \varepsilon_0$. For the limit *not* to exist, this must hold for all L.

   (b) **(5 pts.)** $\lim_{x\to a-} f(x) = L$ – p. 66

2. **(15 pts.)** Show that if $|x_{n+1} - x_n| \leq 2^{-n}$ then x_n is convergent. (Hint: show that it is a Cauchy sequence.)

   Solution. First, note that this chain of inequalities holds:

$$
\begin{aligned}
|x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n| \\
&\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
&\leq 2^{-m+1} + 2^{-m+2} + \cdots + 2^{-n} \text{ (assumption)} \\
&\leq 2^{-n}(1 + 2^{-1} + 2^{-2} + \cdots 2^{-(m-n-1)}) \\
&\leq 2^{-n}\frac{1 - 2^{-(m-n)}}{1/2} = 2(2^{-n} - 2^{-m}) \text{ (geometric series)}
\end{aligned}
$$

   For $\varepsilon/4 > 0$, choose $N \in \mathbb{N}$ such that $2^{-n} < \varepsilon/4$ when $n \geq N$. It follows form the last inequality if $n, m \geq N$ we have that $|x_m - x_n| \leq 2(2^{-n} - 2^{-m}) < 2(2^{-n} + 2^{-m}) < \varepsilon$. Hence, $\{x_n\}$ is a Cauchy sequence and is therefore convergent.

3. **(10 pts.)** Show: $f \vee g(x) := \max\{f(x), g(x)\} = \frac{(f+g)(x) + |(f-g)(x)|}{2}$.

   Solution. Assume $f(x) \geq g(x)$, so $f \vee g(x) = f(x)$. Then, we also have $|(f-g)(x)| = f(x) - g(x)$, and so

$$
\begin{aligned}
\frac{(f+g)(x) + |(f-g)(x)|}{2} &= \frac{f(x) + g(x) + f(x) - g(x)}{2} \\
&= f(x) = f \vee g(x)
\end{aligned}
$$

4. **(15 pts.) (Sequential Characterization of Limits)** Prove this: Let $a \in I \subseteq \mathbb{R}$, where I is open, and let $f : I \setminus \{a\} \to \mathbb{R}$. If $f(x_n) \to L$ for every sequence $x_n \in I \setminus \{a\}$ such that $x_n \to a$, then $L = \lim_{x \to a} f(x)$ exists.

   Proof. Suppose not. Then, for some $\varepsilon_0 > 0$ and every δ there is an x such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \varepsilon_0$. Take $\delta = 1, 1/2, \ldots, 1/n, \ldots$. for each choice of δ, we have x_n such that $0 < |x_n - a| < 1/n$ and $|f(x_n) - L| \geq \varepsilon_0$. By the squeeze theorem for sequences, $x_n \to a$. Thus, from our assumption, $f(x_n) \to L$. Equivalently, $\lim_{n\to\infty} |f(x_n) - L| = 0$. However, by the comparison theorem, $\lim_{n\to\infty} |f(x_n) - L| \geq \varepsilon_0 > 0$. This is a contradiction, so $\lim_{x\to a} f(x) = L$.