Notes on a 1D-Sobolev Theorem

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1 Background

We have been working with distributions. Part of our purpose here is to show that if a distribution u has a (distributional) derivative $u' \in L^2[a,b]$, then u can be represented by a function in $C[a,b]$. To prove this requires that we prove a result important in its own right, which is version of a theorem of Sobolev. In the one-dimensional case, this is the following "regularity" result.

Theorem 1.1 (Sobolev) *If u is a distribution having a distributional derivative u' in $L^2[a,b]$, then $u \in C[a,b]$ and there is a constant $C > 0$ that depends only on $b-a$ for which*

$$\|u\|_{C[a,b]} \le C\|u\|_{H^1[a,b]}, \quad \|u\|^2_{H^1[a,b]} := \int_a^b (|u|^2 + |u'|^2)dx \qquad (1)$$

2 Inequalities

A typical method for proving regularity theorems is to first prove inequalities relating norms on different spaces, assuming that the function is smooth. One can also do this more directly, via properties of the Lebesgue integral. The regularity theorem is then obtained by a density argument. In this section we will carry out this first step in several lemmas.

Lemma 2.1 *Let $v \in C[a,b]$ and let u be any antiderivative for v. If $\delta > 0$ and $\omega(u,\delta)$ is the modulus of continuity for u, then ω satisfies this inequality,*

$$\omega(u,\delta) \le \sqrt{\delta}\|v\|_{L^2}\,. \qquad (2)$$

In addition, there is a constant $C > 0$ that depends only on $b-a$ for which

$$\|u\|_{C[a,b]} \le C\|u\|_{H^1[a,b]}\,. \tag{3}$$

Proof: For every pair $x, y \in [a,b]$ we have $u(x) - u(y) = \int_y^x v(t)dt$. Consequently, by Schwarz's inequality we see that

$$|u(x) - u(y)| \le \int_a^b |v(t)|dt \le \sqrt{|x-y| \int_y^x |v(t)|^2 dt} \le \sqrt{|x-y|}\|v\|_{L^2}.$$

Taking the supremum over $|x - y| \le \delta$ yields (2).

The integral version of the mean value theorem implies that there is point $c \in (a,b)$ such that $u(c) = \frac{1}{b-a}\int_a^b u(t)dt$. By (2) with $\delta = b-a$ we have $|u(x) - u(c)| \le \omega(u, \sqrt{b-a}\,) \le \sqrt{b-a}\|v\|_{L^2}$. This implies that $|u(x)| \le |u(c)| + \sqrt{b-a}\|v\|_{L^2}$. Since $u(c) = \frac{1}{b-a}\int_a^b u(t)dt$, we can apply Schwarz's inequality to obtain $|u(c)| \le \frac{1}{\sqrt{b-a}}\|u\|_{L^2}$. Putting this together with the previous inequality yields

$$\begin{aligned}
|u(x)| &\le \frac{1}{\sqrt{b-a}}\|u\|_{L^2} + \sqrt{b-a}\|v\|_{L^2} \\
&\le \sqrt{\frac{2}{b-a}\|u\|_{L^2}^2 + 2(b-a)\|v\|_{L^2}^2} \\
&\le \underbrace{\sqrt{2\max\{(b-a)^{-1}, b-a\}}}_{C}\,\|u\|_{H^1}
\end{aligned}$$

Again, taking the supremum over $x \in [a,b]$ above yields (3) ∎

The next lemma is a simple corollary of the previous one.

Lemma 2.2 *If $v \in C[a,b]$ and if $U(x) := \int_a^x v(t)dt$, with $a \le x \le b$, then*

$$\|U\|_{L^2[a,b]} \le \frac{b-a}{\sqrt{2}}\|v\|_{L^2}\,. \tag{4}$$

Proof: Schwarz's inequality implies that $|U(x)|^2 \le (x-a)\|v\|_{L^2}^2$. We obtain (4) by integrating both sides from $x = a$ to $x = b$ and then taking square roots of the result. ∎

3 Proof of the 1D-Sobolev Theorem

We suppose that $u' \in L^2[a,b]$ is the derivative of a distribution u. Because $C[a,b]$ is dense in $L^2[a,b]$, there is a sequence of functions $\{v_n \in C[a,b]\}_{n=1}^{\infty}$ such that $v_n \to u'$ in L^2. Let $U_n(x) := \int_a^x v_n(t)dt$, where $a \le x \le b$. Applying Lemma 2.2 to $U_n - U_m$, we have that

$$\|U_n - U_m\|_{L^2[a,b]} \le \frac{b-a}{\sqrt{2}} \|v_n - v_m\|_{L^2} .$$

Since v_n is convergent in L^2, it is a Cauchy sequence. The inequality above implies that U_n is also a Cauchy sequence in L^2 and is therefore convergent to $U \in L^2$.

Now we apply (2) of Lemma 2.1, with $u = U_n - U_m$, $v = v_n - v_m$, and $\delta = b - a$, to obtain

$$|(U_n - U_m)(x) - \underbrace{(U_n - U_m)(0)}_{0}| \le \omega(U_n - U_m, b-a) \le \sqrt{b-a}\|v_n - v_m\|_{L^2} ,$$

from which it follows that

$$\|U_n - U_m\|_{C[a,b]} \le \sqrt{b-a}\|v_n - v_m\|_{L^2} ,$$

and so $\{U_n\}$ is a Cauchy sequence in $C[a,b]$; therefore it converges in $C[a,b]$ to U. Hence, $U \in C[a,b]$.

Let ϕ be in $\mathcal{D}$ and suppose $\text{supp}\{\phi\} \subseteq [a,b]$. Because $U_n' = v_n$, we have that $\langle v_n, \phi\rangle = \langle U_n', \phi\rangle = -\langle U_n, \phi'\rangle$. The L^2 convergence of the sequences v_n and U_n is enough to imply that $\langle v_n, \phi\rangle \to \langle u', \phi\rangle$ and $\langle U_n, \phi'\rangle \to \langle U, \phi'\rangle$. We then have $\langle U, \phi'\rangle = -\langle u', \phi\rangle$.

As is the case in ordinary calculus, if two distributions have the same distributional derivatives, then they differ by a constant. In particular, if u and U have u' as a derivative, then there is a constant k such that $u = k + U$. Let $u_n = k + U_n$ and note that $u_n \to u$ in $C[a,b]$ and L^2. Moreover, $u_n' = U_n' = v_n$. Use these in (3) and let $n \to \infty$ to prove (1).