Spherical Harmonics on $\mathbf{S}^2$

1 The Laplace-Beltrami Operator

In what follows, we describe points on $\mathbf{S}^2$ using the parametrization

$$x = \cos\varphi \sin\theta,\ \ y = \sin\varphi\sin\theta,\ \ z = \cos\theta,$$

where θ is the colatitude and φ is the azimuthal angle. There are coordinate singularities at the north and south poles and along the median $\varphi = 0, 2\pi$. Thus, for a continuous function $f(\theta,\varphi)$ to be well defined on $\mathbf{S}^2$, it must satisfy the following boundary conditions:

- $f(\theta,\varphi) = f(\theta,\varphi+2\pi)$
- $f(0,\varphi)$ and $f(\pi,\varphi)$ are independent of φ.

In these coordinates, the invariant area element on $\mathbf{S}^2$ is given by $d\mu = \sin\theta d\theta d\varphi$. If f is defined and continuous on $\mathbf{S}^2$, then its integral over $\mathbf{S}^2$ is

$$\int_{\mathbf{S}^2} f d\mu = \int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)\sin\theta d\theta d\varphi.$$

The space of functions satisfying $\int_{\mathbf{S}^2}|f|^2 d\mu < \infty$ is denoted by $L^2(\mathbf{S}^2)$. If we define the usual inner product and norm on L^2,

$$\langle f, g\rangle := \int_{\mathbf{S}^2} f\bar{g} d\mu \text{ and } \|f\| := \left\{\int_{\mathbf{S}^2}|f|^2 d\mu\right\}^{\frac{1}{2}},$$

then it is a Hilbert space. The Laplace Beltrami operator on $\mathbf{S}^2$ is given by

$$\Delta_S = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}\right\} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

This operator places the same rôle as the ordinary Laplacian in Euclidean space, and it satisfies properties similar to its Euclidean counterpart. All of those listed below are established via integration by parts for functions f, g satisfying the necessary boundary conditions.

1. Δ_S is selfadjoint: $\langle \Delta_S f, g\rangle = \langle f, \Delta_S g\rangle$.
2. Symmetric form: $\langle \Delta_S f, g\rangle = -\int_{\mathbf{S}^2}\left(\frac{\partial f}{\partial\theta}\frac{\partial\bar{g}}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial f}{\partial\varphi}\frac{\partial\bar{g}}{\partial\varphi}\right) d\mu$
3. Non-negativity of $-\Delta_S$: $-\langle \Delta_S f, f\rangle = \int_{\mathbf{S}^2}\left(\left|\frac{\partial f}{\partial\theta}\right|^2 + \frac{1}{\sin^2\theta}\left|\frac{\partial f}{\partial\varphi}\right|^2\right) d\mu \geq 0.$

2 The Eigenvalue Problem

We now turn to the eigenvalue problem for Δ_S. We seek to find all eigenvalues λ for which there is a non-trivial eigenfunction Y such that

$$\Delta_S Y + \lambda Y = 0.$$

From the properties of Δ_S we have the following theorem.

Theorem 2.1 *The eigenvalues λ are all real, non negative, and the eigenfunctions corresponding to distinct eigenvalues are orthogonal in the inner product of $L^2(\mathbf{S}^2)$. In addition, if Y is an eigenfunction corresponding to λ, we have*

$$\lambda = \frac{1}{\|Y\|^2} \int_{\mathbf{S}^2} \left(\left|\frac{\partial Y}{\partial \theta}\right|^2 + \frac{1}{\sin^2\theta} \left|\frac{\partial Y}{\partial \varphi}\right|^2 \right) d\mu \tag{1}$$

Proof. The expression for λ in (1) follows from properties 2 and 3 for Δ_S, with both f and g being taken as Y. This of course implies that the eigenvalues are real and nonnegative. To obtain orthogonality, use property 1 with $f = Y_1$ and $g = Y_2$, where Y_1, Y_2 are eigenfunctions corresponding to $\lambda_1 \neq \lambda_2$. Then, we have

$$\begin{aligned} \langle \Delta_S Y_1, Y_2 \rangle &= \langle Y_1, \Delta_S Y_2 \rangle \\ \langle \lambda_1 Y_1, Y_2 \rangle &= \langle Y_1, \lambda_2 Y_2 \rangle \\ \lambda_1 \langle Y_1, Y_2 \rangle &= \bar{\lambda}_2 \langle Y_1, Y_2 \rangle \end{aligned}$$

Now, since the eigenvalues are real, $\bar{\lambda}_2 = \lambda_2$. Hence, $(\lambda_1 - \lambda_2)\langle Y_1, Y_2\rangle = 0$, and, because $\lambda_1 \neq \lambda_2$, $\langle Y_1, Y_2 \rangle = 0$. □

From (1), it is easy to see that $\lambda = 0$ is in fact an eigenvalue, and that the only eigenfunction corresponding to it is $Y =$ constant. So assume that $\lambda > 0$ for the rest of the discussion, and that that Y is an eigenfunction corresponding to λ.

The function Y satisfies the boundary conditions described earlier; thus, Y is 2π-periodic in φ. If we fix θ, then we can expand $Y(\theta, \varphi)$ in a Fourier series in φ,

$$Y(\theta, \varphi) = \sum_{m=-\infty}^{\infty} Y_m(\theta) e^{im\varphi}. \tag{2}$$

Apply Δ_S to both sides and assume that switching sum and Δ_S is permissible. The result is

$$\begin{aligned} -\lambda Y = \Delta_S Y &= \sum_{m=-\infty}^{\infty} \Delta_S Y_m(\theta) e^{im\varphi} \\ \sum_{m=-\infty}^{\infty} (-\lambda) Y_m e^{im\varphi} &= \sum_{m=-\infty}^{\infty} \left(\frac{1}{\sin\theta} \frac{d}{d\theta} \left\{ \sin\theta \frac{dY_m}{d\theta} \right\} - \frac{m^2}{\sin^2\theta} Y_m \right) e^{im\varphi} \end{aligned}$$

Comparing Fourier coefficients in the Fourier series for Y and the Fourier series for λY above, we see that

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left\{ \sin\theta \frac{dY_m}{d\theta} \right\} - \frac{m^2}{\sin^2\theta} Y_m = -\lambda Y_m \tag{3}$$

In addition, we have shown that

$$\Delta_S (Y_m e^{im\varphi}) + \lambda Y_m e^{im\varphi} = 0 \tag{4}$$

In other words, each non-zero component of the Fourier series for Y is also an eigenfunction for Δ_S corresponding to λ. (Functions that are identically 0 are *not* called eigenfunctions.) Consequently, we may use $Y_m e^{im\varphi}$ as Y in (1). This gives

$$\begin{aligned} \lambda &= \frac{1}{\|Y_m e^{im\varphi}\|^2} \int_{\mathbf{S}^2} \left(\left| \frac{\partial Y_m e^{im\varphi}}{\partial\theta} \right|^2 + \frac{1}{\sin^2\theta} \left| \frac{\partial Y_m e^{im\varphi}}{\partial\varphi} \right|^2 \right) d\mu \\ &\geq \frac{1}{\|Y_m\|^2} \int_{\mathbf{S}^2} \int_{\mathbf{S}^2} \frac{1}{\sin^2\theta} \left| \frac{\partial Y_m e^{im\varphi}}{\partial\varphi} \right|^2 d\mu \\ &\geq \frac{m^2}{\|Y_m\|^2} \int_{\mathbf{S}^2} \frac{1}{\sin^2\theta} \left| Y_m \right|^2 d\mu \\ &\geq \frac{m^2}{\|Y_m\|^2} \|Y_m\|^2 = m^2 \end{aligned}$$

This puts an upper bound on the number of Fourier components in Y. Namely, the largest value of $|m|$ is the integer $\ell := \lfloor \sqrt{\lambda} \rfloor$, and also on the dimension of the eigenspace of λ. This is because Y_m satisfies the ODE (3), which has at most two linearly independent solutions. In fact, the boundary conditions for functions on $\mathbf{S}^2$ allow only one solution for given integer $|m|$.

Since m itself runs from $m = -\ell$ to $m = \ell$, there are only $2\ell + 1$ solutions. Thus the dimension of the eigenspace corresponding to λ is $2\ell+1 = 2\lfloor\sqrt{\lambda}\rfloor+1$. We summarize this below.

Proposition 2.2 *For $\lambda > 0$, the eigenfunctions $\Delta_S Y + \lambda Y$ are linear combinations of solutions to (4), where Y_m is, up to a constant, uniquely determined by (3). Moreover, if $\ell := \lfloor\sqrt{\lambda}\rfloor$, then $|m| \leq \ell$ and the dimension of the eigenspace of λ is $2\ell + 1$.*

The next step in solving the eigenvalue problem is to introduce two new operators,

$$L_{\pm} = e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\varphi}\right). \tag{5}$$

For reasons that will be clear later, L_+ is the called the *raising operator* and L_- is the *lowering operator*. A routine, if tedious, calculation gives that the raising and lowering operators commute with Δ_S – that is,

$$L_{\pm}\Delta_S = \Delta_S L_{\pm}\,.$$

Consequently, if $\Delta_S Y = -\lambda Y$, we have that $L_{\pm}\Delta_S Y = \Delta_S L_{\pm} Y = -\lambda L_{\pm} Y$. Thus, $L_{\pm}Y$ are also eigenfunctions of Δ_S, assuming they are not zero. In particular, we can apply them to $Y_m(\theta)e^{im\varphi}$,

$$L_{\pm}\left(Y_m(\theta)e^{im\varphi}\right) = \left(\frac{dY_m}{d\theta} \mp m\cot\theta Y_m(\theta)\right)e^{i(m\pm1)\varphi}.$$

The right side above is an eigenfunction that has its Fourier series in φ consisting of a single term. By what we said above, it must be a multiple of $Y_{m\pm1}(\theta)e^{i(m\pm1)\varphi}$. Now, let $m = \ell$ and use L_+. The result would be an eigenfunction $Y_{\ell+1}(\theta)e^{i(\ell+1)\varphi}$. That is, an eigenfuntion with $m = \ell + 1 > \ell$. The previous proposition implies that the only way this is possible is if

$$L_+\left(Y_\ell(\theta)e^{i\ell\varphi}\right) = \left(\frac{dY_\ell}{d\theta} - \ell\cot\theta Y_\ell(\theta)\right)e^{i(\ell+1)\varphi} = 0,$$

which implies that Y_ℓ satisfies the first order ODE, $\frac{dY_\ell}{d\theta} - \ell\cot\theta Y_\ell(\theta) = 0$. Solving this is easy; the result is $Y_\ell(\theta) = C\sin^\ell\theta$, $C \neq 0$. Plugging this solution back into (3) results in the identity $-\ell(\ell+1)Y_\ell = -\lambda Y_\ell$, from which it follows that $\lambda = \ell(\ell+1)$. The other eigenfunctions for $\lambda = \ell(\ell+1)$ may be found by recursively applying L_- to $Y_\ell(\theta)e^{i\ell\varphi}$. Up to normalization, this procedure gives us the eigenvalues and eigenfunctions of Δ_S. We have thus proved the following.

Theorem 2.3 *The eigenvalues for $\Delta_S Y + \lambda Y$ are of the form $\lambda = \ell(\ell+1)$, where $\ell = 0, 1, 2, \ldots$. Corresponding to each ℓ, there are $2\ell + 1$ linearly independent eigenfunctions, $Y_{\ell,m}(\theta, \varphi) = L_-^{\ell-m}\left(\sin^\ell \theta e^{i\ell\varphi}\right)$.*

3 The Spherical Harmonics

Spherical harmonics are eigenfunctions of the Laplace-Beltrami operator Δ_S. Theorem 2.1 tells us that spherical harmonics corresponding to differing values of ℓ and, hence differing values of $\lambda = \ell(\ell+1)$, are orthogonal. Theorem 2.3 gives us a way of constructing a basis of spherical harmonics for each fixed ℓ. Because of the factor $e^{im\varphi}$ in each of these, they are also orthogonal. By adjusting normalization constants, one can get the all of the spherical harmonics to be an orthonormal set.

Theorem 3.1 *For $\ell = 0, 1, 2, \ldots$, and $m = -\ell, \ldots, \ell$, the functions*

$$Y_{\ell,m}(\theta, \phi) := \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}} \sin^{|m|} \theta \, P_\ell^{(|m|)}(\cos\theta) e^{im\varphi}$$

form an orthonormal set. Moreover, these are a basis for $L^2(\mathbf{S}^2)$. The function $P_\ell(x)$ whose derivative appears above is the ℓ-th order Legendre polynomial.

We remark that the normalization for $Y_{\ell,m}$ is only one of many possible. Also, there are real versions of the spherical harmonics that use $\sin(m\varphi)$ and $\cos(m\varphi)$ rather than $e^{im\varphi}$.