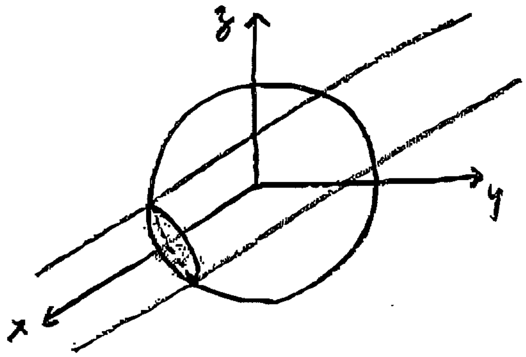
Math 251 Fall 2005 Final Exam Sample

There are a total of eight problems. No calculators are allowed.

1. (a) Use Stokes' Theorem to compute the integral $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = yz\,\vec{i} - xz\,\vec{j} + (z^3 - xy + y)\,\vec{k}$ and the surface S is the part of the sphere $x^2+y^2+z^2 = a^2$ that lies inside the cylinder $y^2+z^2 = b^2$ and satisfies $x > 0$. Note that $a > b > 0$. (10%)

   (b) Give a sketch of the surface and region under consideration in (a). (3%)

Soln Stokes' Theorem gives $\iint_S \nabla\times\vec{F}\cdot d\vec{S} = \oint_{\partial S} \vec{F}\cdot d\vec{r}$, where S and ∂S can be seen from the sketch below:



On ∂S, $x^2+y^2+z^2 = x^2+(y^2+z^2) = x^2+b^2 = a^2$, so $x = \sqrt{a^2-b^2}$ (>0).

The curve ∂S lies on $y^2+z^2=b^2$. So we parametrize it by

$$\left.\begin{array}{l} y = b\cos\theta \\ z = b\sin\theta \end{array}\right\} \quad 0 \le \theta \le 2\pi :$$

$$\oint_{\partial S} \vec{F}\cdot d\vec{r} = \int_{\partial S} (yz)\,dx + (-xz)\,dy + (z^3 - xy + y)\,dz$$

$$= \int_0^{2\pi} \Big\{ b\cos\theta \cdot b\sin\theta \, d(\sqrt{a^2-b^2}) - \sqrt{a^2-b^2}\, b\sin\theta \, d(b\cos\theta) + (b^3\sin^3\theta - \sqrt{a^2-b^2}\cdot b\cos\theta + b\cos\theta)\, d(b\sin\theta)$$

$$= \int_0^{2\pi} \Big\{ 0 + b^2\sqrt{a^2-b^2}\sin^2\theta \, d\theta + (b^4 \sin^3\theta - b^2\sqrt{a^2-b^2}\cos^2\theta + b^2\cos^2\theta)\, d\theta \Big\}$$

$$= \pi b^2\sqrt{a^2-b^2} + b^2(1 - \sqrt{a^2-b^2})\pi = \pi b^2.$$

2. Selected parts from
(a) Exercise 20, p.931 and
(b) Exercises 21–26, p.937.

3. Find the flux of the vector field $\underline{F}(x,y,z) = (5x^2 - z)\,\underline{i} + 6y\,\underline{j}$ $-(x-6z)\,\underline{k}$ over the sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$, where $r > 0$.
(Gauss Divergence Theorem)

Sol'n Flux $\iint\limits_S \vec{F} \cdot d\vec{S} = \iiint\limits_V \nabla \cdot \vec{F}\, dV$.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(5x^2 - z) + \frac{\partial}{\partial y}(6y) + \frac{\partial}{\partial z}[-(x-6z)] = 10x + 6 + 6 = 10x + 12$$

The ball V can be parametrized by

$$\left.\begin{aligned} x &= a + \rho \sin\phi \cos\theta \\ y &= b + \rho \sin\phi \sin\theta \\ z &= c + \rho \cos\phi \end{aligned}\right\} \quad \begin{aligned} &0 \le \phi \le \pi, \\ &0 \le \theta \le 2\pi, \\ &0 \le \rho \le r \end{aligned} \qquad dV = \underbrace{\rho^2 \sin\phi\, d\phi\, d\theta\, d\rho}_{\text{You can take this for granted!}}$$

So $$\iiint\limits_V (10x + 12)\, dV = \iiint\limits_V [10(x - a + a) + 12]\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\rho} [10\rho \sin\phi \cos\theta + (10a + 12)]\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 10 \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{0} \int_0^{\pi} \sin^2\phi\, d\phi \int_0^{\rho} \rho^3\, d\rho + (10a + 12) \iiint\limits_V dV$$

$= (10a + 12) \cdot \underbrace{\text{volume of the ball}}_{}$

$= (10a + 12) \cdot \dfrac{4}{3} \pi r^3$.

4. Exercise 23, p. 899. (Green's Theorem)
(Already discussed in Class).

5. To be selected from Test I, Problems 1 and 4:
Problem 1 is the Laplace equation;
Problem 4 is the limit.
See the Answer Key posted.

6. To be selected from Test II, Problems 1 and 2:
Problem 1 is the constrained max and min problem;
Problem 2 is the critical point problem.
See the Answer Key posted.

7. Problem 4, Test II:

$$\int_{-2}^{2}\int_{-2}^{2y+2} \sin(-x^2+12x+5)\,dx\,dy \quad \cdots$$

8. To be selected from Test III, Problems 1 and 5:
Problem 1 involves finding f such that $\vec{F} = \nabla f$;
Problem 5 involves finding the volume of an ice cream cone.
See the Answer Key posted.