There are a total of five problems.
No calculators are allowed.

1. Let \( A \) be an \( m \times n \) matrix.

   (i) Define the left and right inverses of \( A \). (5%)
   (ii) Show that if the column vectors of \( A \) span \( \mathbb{R}^m \), then \( A \) has a right inverse. (5%)
   (iii) Show that if the column vectors of \( A \) are linearly independent, then \( A \) has a left inverse. (5%)
   (iv) Show that \( m = n \) if \( A \) satisfies the assumptions in both parts (i) and (ii) above? (5%)

Solution:
(i) A matrix \( L \) of dimension \( n \times m \) satisfying \( LA = I_n \) is called a left inverse of \( A \). A matrix \( R \) of dimension \( n \times m \) satisfying \( AR = I_m \) is called a right inverse of \( A \).

(ii) Let \( A = [\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n] \), where \( \hat{a}_j, 1 \leq j \leq n \), are the column vectors. Then because \( \hat{a}_1, \ldots, \hat{a}_n \) span \( \mathbb{R}^m \), the standard basis vectors \( \hat{e}_1, \hat{e}_2, \ldots, \hat{e}_m \) can be expressed as linear combinations of \( \hat{a}_1, \ldots, \hat{a}_n \) as follows:

\[
\hat{e}_j = \hat{a}_1 k_{1j} + \hat{a}_2 k_{2j} + \ldots + \hat{a}_n k_{nj}, \quad j = 1, 2, \ldots, m.
\]

That is,

\[
\begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1m} \\ k_{21} & k_{22} & \cdots & k_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nm} \end{pmatrix} \equiv AK = [\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_m] = I_m.
\]

Thus, \( K \) is a right inverse of \( A \).

(iii) If the column vectors of \( A \) are linearly independent, then \( \text{rank} A = n \). Since "row rank = column rank", the number of linearly independent rows is also equal to \( n \).

Now consider \( A^T \). The column vectors of \( A^T \), which are row vectors of \( A \), have rank \( n \) in \( \mathbb{R}^n \). Thus these column vectors span \( \mathbb{R}^n \). We can apply part (ii) to show that \( A^T \) has a right inverse,

\[ A^T G = I_n, \text{ for some } m \times n \text{ matrix } G. \]

So \( (A^T G)^T = I_n^T = I_m \), \( GA^T = I_n^1 \). Hence \( G^T \) is a left inverse of \( A \).
2. Let \( L: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) be a linear transformation defined by
\[
L(\vec{x}) = (x_1 + 2x_2 - x_3, x_1 - 2x_2 + x_3), \text{ for } \vec{x} = (x_1, x_2, x_3)^T.
\]

(i) Find the matrix representation of \( L \) with respect to the ordered bases \([\vec{v}_1, \vec{v}_2, \vec{v}_3]\) and \([\vec{w}_1, \vec{w}_2]\), where
\[
\vec{v}_1 = (1, 0, 0)^T, \quad \vec{v}_2 = (1, 1, 0)^T, \quad \vec{v}_3 = (1, 1, 1)^T
\]
and
\[
\vec{w}_1 = (1, 2)^T, \quad \vec{w}_2 = (3, 1)^T
\] (15%) 

(ii) State the theorem based on which you have used to solve the problem in part (i). (5%) 

\[ \text{Solution: (i) Write the matrix} \]

\[
\begin{bmatrix}
\vec{w}_1 & \left( \overrightarrow{L(\vec{v}_1)} \right) & \left( \overrightarrow{L(\vec{v}_2)} \right) & \left( \overrightarrow{L(\vec{v}_3)} \right)
\end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 2 \end{bmatrix} R_1 \begin{bmatrix} 2 & 1 & 1 & -1 \end{bmatrix} R_2^{-2} R_1
\]

Perform Gaussian elimination as shown above \( \Rightarrow \)
\[
\begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & -5 & -1 & -1 \end{bmatrix} R_2 \Rightarrow \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} R_2^{-2} R_1 \Rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}
\]

The matrix \( A \) shown above is the representation.

(ii) Let \( L: \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation. Let \( \{\vec{v}_1, \cdots, \vec{v}_n\} \) be a basis for \( \mathbb{R}^n \) and \( \{\vec{w}_1, \cdots, \vec{w}_m\} \) be a basis for \( \mathbb{R}^m \). Then the matrix representation \( A \) for \( L \) with respect to the above ordered bases is obtainable from
\[
\begin{bmatrix} \vec{w}_1 & \cdots & \vec{w}_m \\ \overrightarrow{L(\vec{v}_1)} & \cdots & \overrightarrow{L(\vec{v}_n)} \end{bmatrix} \xrightarrow{\text{rowechelon form}} \begin{bmatrix} \text{rowechelon form} \\ \text{form} \end{bmatrix} \Rightarrow \begin{bmatrix} \text{Im} & A \end{bmatrix}.
\]
3. Let \( S \) be the subspace of \( C[0,1] \) spanned by \( 1, e^x \) and \( xe^x \). Define the linear operator \( L \) on \( S \) by
\[
L(f(x)) = f'(x) - f(1).
\]
Find the matrix representing \( L \) with respect to the ordered basis \([1, e^x, xe^x]\). (20%)

**Solution**

\[
L(1) = 1' - 1 = -1
\]

\[
L(e^x) = (e^x)' - e^1 = -e^2 + e^x = -e + e^x
\]

\[
L(xe^x) = (xe^x)' - 1 \cdot e^1 = -e + \left[ e^x + xe^x \right] = -e + e^x + xe^x.
\]

Thus the matrix representation of \( L \) is
\[
\begin{bmatrix}
-1 & -e & -e \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}.
\]
4. Let $A$ be an $m \times n$ matrix. Show that

(i) If $\vec{x} \in N(A^T A)$, then $A\vec{x} \in R(A) \cap N(A^T)$. (5%)

(ii) $N(A^T A) = N(A)$. (5%)

(iii) $A$ and $A^T A$ have the same rank. (5%)

(iv) If $A$ has linearly independent columns, then $A^T A$ is nonsingular. (5%)

Solution:

(i) If $\vec{x} \in N(A^T A)$, then $(A^T A)\vec{x} = \vec{0}$; so $A^T (A^T A)\vec{x} = \vec{0}$, i.e., $A^T \vec{x} \in N(A^T)$. Also, $A^T \vec{x} \in R(A)$ by the very definition of $R(A)$. Hence $A^T \vec{x} \in N(A^T) \cap R(A)$.

(ii) We first show $N(A) \subseteq N(A^T A)$. If $\vec{x} \in N(A)$, then $A\vec{x} = \vec{0}$, so $A^T (A\vec{x}) = \vec{0}$. This gives $(A^T A)\vec{x} = \vec{0}$, i.e., $\vec{x} \in N(A^T A)$.

Next, we show $N(A^T A) \subseteq N(A)$. If $\vec{x} \in N(A^T A)$, then $A^T A \vec{x} = \vec{0}$. Therefore $0 = A^T A \vec{x} = A^T (A\vec{x}) = \vec{a}^T \vec{a}$. This gives $A\vec{x} = \vec{0}$. Hence $\vec{x} \in N(A)$.

(iii) Since

\[ \text{rank } A + \text{nullity } A = n, \]

\[ \text{rank } A^T A + \text{nullity } A^T A = n, \]

by part (ii) we have nullity $A = \text{nullity } A^T A$. Hence rank $A = \text{rank } A^T A$.

(iv) $A^T A$ is an $n \times n$ (square) matrix and, by assumption, rank $A = n$.

Use part (iii); we see that nullity $A^T A = \text{nullity } A = 0$. Therefore $A^T A$ is an invertible matrix because $A^T A$ doesn't have any nontrivial solution.
5. Let $A$ be an $m \times n$ matrix of rank $n$. Show that $\hat{x} = (A^T A)^{-1} A^T b$ is the unique least squares solution to $Ax = b$. (20%)

Solution: Since $\mathbb{R}^m = \text{R}(A) \oplus [\text{R}(A)]^\perp$, we write

$$b = \hat{b}_1 + \hat{b}_2,$$

where $\hat{b}_1 \in \text{R}(A)$ and $\hat{b}_2 \in \text{R}(A)^\perp$.

$\hat{x}$ is a solution to the least square problem

$$\min_{\hat{x} \in \mathbb{R}^m} |A\hat{x} - \hat{b}|^2 = |A\hat{x} - \hat{b}_1|^2$$

iff

$$A\hat{x} = \hat{b}_1$$

because otherwise $A\hat{x} - \hat{b}_1 \neq \hat{b}_2$ and

$$|A\hat{x} - \hat{b}|^2 = |A\hat{x} - (\hat{b}_1 + \hat{b}_2)|^2 = |A\hat{x} - \hat{b}_1|^2 + |\hat{b}_2|^2$$

$$> |\hat{b}_2|^2 = \min_{\hat{x} \in \mathbb{R}^m} |A\hat{x} - \hat{b}|^2,$$

which violates the least square property of $\hat{x}$.

But $A\hat{x} = \hat{b}_1 = \hat{b} - \hat{b}_2$ is equivalent to

$$A\hat{x} - \hat{b} = -\hat{b}_2 \in \text{R}(A)^\perp = \text{N}(A^T).$$

Thus

$$A^T(A\hat{x} - \hat{b}) = 0$$

$$A^T A \hat{x} = A^T \hat{b}, \quad \hat{x} = (A^T A)^{-1} A^T \hat{b}.$$  

Note that $A^T A$ is invertible by problem 4(v) of this test.

Also, $\hat{x}$ is unique because $\hat{x}$ is given explicitly as

$$\hat{x} = (A^T A)^{-1} A^T \hat{b}. $$