TENSORS: GEOMETRY AND APPLICATIONS
CORRECTIONS AND ADDITIONS, LAST UPDATED 7/13

J.M. LANDSBERG

1. Chapter 1

- p7, displayed equation for A, third term, change $a_1^2e^2\otimes f_1$ to $a_1^2e^1\otimes f_2$
- p7, Exercise1.2.1.1 delete XS.
- p32, §2.3.1, line 4, $W^*\to V$ change to $W^*\to V^*$.
- p36, indices in (2.4.3) should be: 3rd, 4th terms $c_1^2\to c_2^1$, 5th, 6th, $c_2^1\to c_1^2$.

2. Chapter 2

- p46, Exercise 2.6.1.2 (1), should say $f:V\to W$ with $\dim W\geq \dim V$, and (3) should clarify $f:V\to V$.
- p49, last displayed equation, first line, missing $\otimes$ two times.
- p51, Eqn. (2.7.1) denominator $2^n n!$ change to $2^m m!$ and 2^n change to 2^m.
- p53, Exercise 2.8.1 (1), add, "in particular, show $\dim S_{21}V=\frac{n(n^2-1)}{3}$".
- p53, Exercise 2.8.1 (8) change "so that we" to "and that we". Also add hint that the exactness is a special case of the Poincare Lemma.
- p65, Fig. 2.11.14: of the six realizations, only three are distinct.

3. Chapter 3

- p70,Thm. 3.1.4.3 reference [220] should be there and in (2), $\mathbf{v}^3-1$ should be $\mathbf{v}^3$.
- p82, lines 1,3: $A_T^{\wedge 2}$ should be $T_A^{\wedge}$ (5 times).
- p82, §3.8.1.4: In Thm, It remains to show that the equations are not identically zero. This was shown in [4].
- p85, It is now known that $\underline{\mathbf{R}}(M_{m,m,m})\geq 2m^2-m$, see [7].
- p88, Thms. 3.9.3.1, 3.9.3.1, add "and Strassen's degree 5 equations."
- p89, first displayed matrix, indices in $(2,3)$ slots of blocks 1,3,4,6 are incorrect. They should respectively be $(122),(122),(012),(012)$.

4. Chapter 4

- p98, add sentence at the end of 4.1: If these questions cause no difficulty, then skep to Chapter 5.
- p100, add the example in the plane of $\{x^2=0\}$. In any degree d we do not have the equation $xy^{d-1}=0$ which is in the ideal of $\{x=0\}$, showing that set theoretic equations are not the same as scheme theoretic equations.
- p101 line above 4.2.3, change r to $r+1$

5. Chapter 5

- p122, 5.3.1, change "number of ways" to "number of parameters worth that"
- p127, Thm 5.5.1.1(4) $\mathbf{ab}-\mathbf{a}-\mathbf{b}-2$ change to $\mathbf{ab}-\mathbf{a}-\mathbf{b}+2$ two times.

6. Chapter 6

- p140, first bullet: "the number of elements in its conjugacy class" change to "the dimension of the module".
- p140, (6.2.3) $+e_{(132)}$ change to $-e_{(132)}$.
- p147, Thm 6.4.5.1: This is really only half the statement of the Double Commutant Thm., even in this special case, see e.g. [9] for the full statement.
- p140, Exercise 6.2.1(1) $v\rho_{\boxed{1}\boxed{2}\boxed{3}} = \rho_{\boxed{1}\boxed{2}\boxed{3}}$ change to $v\rho_{\boxed{1}\boxed{2}\boxed{3}} = \lambda\rho_{\boxed{1}\boxed{2}\boxed{3}}$ for some $\lambda \in \mathbb{C}^*$, and similarly for the rest of the exercises.
- p153, §6.7.1, displayed eqn., $\sum$ change to $\bigoplus$
- p154, Exercise 6.7.1.1: $c^{\nu}_{\pi,\mu}$ change to $c^{\pi}_{\nu,\mu}$, also Hint: Say $|\pi| = k$, expand $(A{\otimes}B)^{\otimes k}$, and reshuffle to have all A's first, then all B's, and consider the subgroup of $\mathfrak{S}_{2k}$ that preserves the shuffling.
- Exercise 6.7.3(2) $\sum$ change to $\bigoplus$
- p160, Exercise 6.8.1.3 $(x_1 \wedge \cdots \wedge x_p)^p$ change to $x_1^p \cdots x_k^p$
- p161, Prop. 6.8.2.1/Exercise 6.8.2.2. What is proved is that a nonzero *weight* vector is a highest weight vector iff $\mathfrak{n}.v = 0$. To do the exercise, need the fact that $B = exp(\mathfrak{b})$.
- p165, Exercise 6.9.2(2). To make the exercise easier, first do the case of the projection $GL(V)/B$ to a next-to minimal parabolic (show the fiber is a $\mathbb{P}^1$).
- p168, Prop. 6.10.4.1, remove the $S_{2d}A$ terms from the ideal.
- p170, Exercise 6.10.6.6 is difficult, as the natural realizations of the modules will not lie in $S^d(A{\otimes}B)$, one must show their projections into $S^d(A{\otimes}B)$ are nonzero.

7. Chapter 7

- p180, $\Pi\mathbf{a}_i - \sum \mathbf{a}_i - n + 1$ change to $\Pi\mathbf{a}_i - \sum \mathbf{a}_i + n - 1$ in Thm 7.3.1.4(1) and above statement of Thm.
- p199, Displayed equation in Remark 7.7.1.2, second entry of second line should read $\Lambda^5 B{\otimes}\Lambda^5(A{\otimes}C)$.
- p213, §8.2.5, a better modern reference is [2]

8. Chapter 8

- p220, Prop. 8.6.1.2 does not appear in [39], and in fact dates back to Hadamard.
- p220, Proof of Prop. 8.6.1.2 last line $\ell_1^\delta \cdots \ell_d^\delta$ should be $(\ell^1)^\delta \cdots (\ell^d)^\delta$.
- p224-5, roles of V and V^* are accidently switched several times.

9. Chapter 9

- p235/6, Prop. 9.3.1 is not correct as stated because $\underline{\mathbf{R}}(xyz) = 4$, not 3 as stated in proof p 236.

- p236, §9.3.2, the symmetric border rank lower bounds via flattenings are easily seen to be $\binom{n}{\lfloor n/2 \rfloor}^2$ in both cases, as the image of flattenings are the appropriate sized minors (resp. sub-permanents) for the determinant (resp. permanent).

10. Chapter 10

- p260, §10.5. A better, and older reference for this section is [1].
- p266, Thm. 10.9.2.1, S^3W change to S^dW.
- p267, In equation (10.10.1) the first term is not needed
- p269, displayed eqn. (4) $a_3\otimes b_1\otimes c_1 + a_1\otimes b_2\otimes c_2 + a_1\otimes b_1\otimes c_2 + a_2\otimes b_3\otimes c_1 + a_2\otimes b_1\otimes c_3$ change to $a_3\otimes b_1\otimes c_1 + a_1\otimes b_2\otimes c_1 + a_1\otimes b_1\otimes c_1 + a_2\otimes b_3\otimes c_1 + a_2\otimes b_1\otimes c_3$
- p281, reference to [3] is old edition

11. Chapter 11

- p275, the current (as of 6/13) world record for the upper bound of the exponent of matrix multiplication is in [10].
- p276, the current (as of 6/13) world record for the lower bound of the border rank (resp. rank) of matrix multiplication is $\underline{\mathbf{R}}(M_{n,n,n}) \geq 2n^2 - n$ in [7]. For rank, the current (as of 6/13) world record is $\mathbf{R}(M_{n,n,n}) \geq 3n^2 - 2\sqrt{2}n^3/2 - 3n$ [8] following work in [5] which showed $\mathbf{R}(M_{n,n,n}) \geq 3n^2 - 4n^{\frac{3}{2}} + 3n$.
- p280, the targets of teh maps M_2, M_3 should respectively be $\begin{pmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix}$.
- p276, Thm. 11.0.2.12, note also that $23 \geq \mathbf{R}(M_{3,3,3})$, due to Laderman.
- p277, Remark: there are three families of lines on $Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$, lines from distinct families intersect. This gives more insight into the algorithm.
- p277, first displayed equation is for operator with $a_1^1 = 0$, second is for operator with $a_2^2 = 0$, in third, the first c_2^1 should be c_1^2.
- p279, Remark: since $T_1 + T_2 : A^* \times B^* \to C$ is surjective, in fact one has $\underline{\mathbf{R}}(T_1 + T_2) = \mathbf{a}_1\mathbf{b}_1 + 1$.
- p280, second and third displayed equations are missing $\otimes$ before many of the γ's.
- p281, Remark: Prop. 11.3.1.1 is a special case of the algebraic Peter-Weyl Thm 13.6.3.
- p281, §11.3.2, last line of paragraph, $S_1{}^{-1}S_3$ change to $S_3{}^{-1}S_1$.
- p284, first line, reference is incorrect, it should be [6]

12. Chapter 12

13. Chapter 13

- p322 proof of 13.4.1.1: $\mathbb{F}_2$ should be $\{0, 1\}$.

14. Chapter 14

15. Chapter 15

16. Chapter 16

- p389, second line of proof, $\subseteq$ change to $=$. First displayed eqn. of proof, add summation over i in first term,

References

1. Josephine H. Chanler, *The invariant theory of the ternary trilinear form*, Duke Math. J. **5** (1939), 552–566. MR 0000221 (1,35e)
2. Jean Dieudonné, *Sur une généralisation du groupe orthogonal à quatre variables*, Arch. Math. **1** (1949), 282–287. MR 0029360 (10,586l)
3. Roe Goodman and Nolan R. Wallach, *Symmetry, representations, and invariants*, Graduate Texts in Mathematics, vol. 255, Springer, Dordrecht, 2009. MR 2522486
4. J. M. Landsberg, *Explicit tensors of border rank at least 2n-1*, preprint arXiv:1209.1664.
5. ———, *New lower bounds for the rank of matrix multiplication*, preprint arXiv:1206.1530.
6. ———, *Geometry and the complexity of matrix multiplication*, Bull. Amer. Math. Soc. (N.S.) **45** (2008), no. 2, 247–284. MR MR2383305 (2009b:68055)
7. J.M. Landsberg and Giorgio Ottaviani, *New lower bounds for the border rank of matrix multiplication*, preprint, arXiv:1112.6007.
8. Alex Massarenti and Emanuele Raviolo, *The rank of $n \times n$ matrix multiplication is at least* $3n^2 - 2\sqrt{2}n^3/2 - 3n$, arXiv:1211.6320.
9. Claudio Procesi, *Lie groups*, Universitext, Springer, New York, 2007, An approach through invariants and representations. MR MR2265844 (2007j:22016)
10. Virginia Williams, *Breaking the coppersimith-winograd barrier*, preprint.

E-mail address: `jml@math.tamu.edu`