Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

Commutators on ℓ_∞

D. Dosev and W. B. Johnson

Abstract

The operators on ℓ_∞ which are commutators are those not of the form $\lambda I + S$ with $\lambda \neq 0$ and S strictly singular.

1. *Introduction*

The commutator of two elements A and B in a Banach algebra is given by

$$[A, B] = AB - BA.$$

A natural problem that arises in the study of derivations on a Banach algebra $\mathcal{A}$ is to classify the commutators in the algebra. Using a result of Wintner([**18**]), who proved that the identity in a unital Banach algebra is not a commutator, with no effort one can also show that no operator of the form $\lambda I + K$, where K belongs to a norm closed ideal $\mathcal{I}(\mathcal{X})$ of $\mathcal{L}(\mathcal{X})$ and $\lambda \neq 0$, is a commutator in the Banach algebra $\mathcal{L}(\mathcal{X})$ of all bounded linear operators on the Banach space $\mathcal{X}$. The latter fact can be easily seen just by observing that the quotient algebra $\mathcal{L}(\mathcal{X})/\mathcal{I}(\mathcal{X})$ also satisfies the conditions of Wintner's theorem.

In 1965 Brown and Pearcy ([**5**]) made a breakthrough by proving that the only operators on ℓ_2 that are not commutators are the ones of the form $\lambda I + K$, where K is compact and $\lambda \neq 0$. Their result suggests what the classification on the other classical sequence spaces might be, and, in 1972, Apostol ([**3**]) proved that every non-commutator on the space ℓ_p for $1 < p < \infty$ is of the form $\lambda I + K$, where K is compact and $\lambda \neq 0$. One year later he proved that the same classification holds in the case of $\mathcal{X} = c_0$ ([**4**]). Apostol proved some partial results on ℓ_1, but only 30 year later was the same classification proved for $\mathcal{X} = \ell_1$ by the first author ([**6**]). Note that if $\mathcal{X} = \ell_p$ $(1 \leq p < \infty)$ or $\mathcal{X} = c_0$, the ideal of compact operators $K(\mathcal{X})$ is the largest proper ideal in $\mathcal{L}(\mathcal{X})$ ([**8**], see also [**17**, Theorem 6.2]). The classification of the commutators on ℓ_p, $1 \leq p < \infty$, and partial results on other spaces suggest the following

Conjecture 1. Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$, $1 \leq p \leq \infty$ or $p = 0$ (we say that such a space admits a Pełczyński decomposition). Assume that $\mathcal{L}(\mathcal{X})$ has a largest ideal $\mathcal{M}$. Then every non-commutator on $\mathcal{X}$ has the form $\lambda I + K$, where $K \in \mathcal{M}$ and $\lambda \neq 0$.

In [**3**] Apostol obtained a partial result regarding the commutators on ℓ_∞. He proved that if $T \in \mathcal{L}(\ell_\infty)$ and there exists a sequence of projections $(P_n)_{n=1}^\infty$ on ℓ_∞ such that $P_n(\ell_\infty) \simeq \ell_\infty$ for $n = 1, 2, \ldots$ and $\|P_n T\| \to 0$ as $n \to \infty$, then T is a commutator. This condition is clearly

---

2000 *Mathematics Subject Classification* 47B47 (primary), 47L20 (secondary).

Research for both authors is supported in part by NSF grant DMS 05-03688.

This article is part of the first author's doctoral dissertation, which is being prepared at Texas A&M University under the direction of the second author.

satisfied if T is a compact operator, but, as the first author showed in [**6**], it is also satisfied if T is strictly singular, which is an essential step for proving the conjecture for ℓ_∞.
In order to give a positive answer to the conjecture one has to prove

– Every operator $T \in \mathcal{M}$ is a commutator
– If $T \in \mathcal{L}(\mathcal{X})$ is not of the form $\lambda I + K$, where $K \in \mathcal{M}$ and $\lambda \neq 0$, then T is a commutator.

In this paper we will give positive answer to this conjecture for the space ℓ_∞.

The authors would like to thank the referee and Dr. Niels Jakob Lausten for their comments and suggestions that help improve the manuscript.

2. *Notation and basic results*

For a Banach space $\mathcal{X}$ denote by the $\mathcal{L}(\mathcal{X})$, $\mathcal{K}(\mathcal{X})$, $\mathcal{C}(\mathcal{X})$ and $S_{\mathcal{X}}$ the space of all bounded linear operators, the ideal of compact operators, the set of all finite co-dimensional subspaces of $\mathcal{X}$ and the unit sphere of $\mathcal{X}$. By *ideal* we always mean closed, non-zero, proper ideal. A map from a Banach space $\mathcal{X}$ to a Banach space $\mathcal{Y}$ is said to be strictly singular if whenever the restriction of T to a subspace M of $\mathcal{X}$ has a continuous inverse, M is finite dimensional. In the case where $\mathcal{X} \equiv \mathcal{Y}$, the set of strictly singular operators forms an ideal which we will denote by $\mathcal{S}(\mathcal{X})$. Recall that for $\mathcal{X} = \ell_p$, $1 \leq p < \infty$, $\mathcal{S}(\mathcal{X}) = \mathcal{K}(X)$ ([**8**]) and on ℓ_∞ the ideals of strictly singular and weakly compact operators coincide ([**1**, Theorem 5.5.1]). A Banach space $\mathcal{X}$ is called *prime* if each infinite-dimensional complemented subspace of $\mathcal{X}$ is isomorphic to $\mathcal{X}$. The spaces ℓ_p, $1 \leq p \leq \infty$, are all prime (cf. [**13**, Theorem 2.a.3 and Theorem 2.a.7]). We say that a linear operator between two Banach spaces $T : \mathcal{X} \to \mathcal{Y}$ is an isomorphism if T is injective bounded linear map. If in addition T is surjective we will say that T is an onto isomorphism. For any two subspaces (possibly not closed) X and Y of a Banach space $\mathcal{Z}$ let

$$d(X, Y) = \inf\{\|x - y\| : x \in S_X,\, y \in Y\}.$$

A well known consequence of the open mapping theorem is that for any two closed subspaces X and Y of $\mathcal{Z}$, $d(X, Y) > 0$ if and only if $X \cap Y = \{0\}$ and $X + Y$ is a closed subspace of $\mathcal{Z}$. Note also that $d(X, Y) = 0$ if and only if $d(Y, X) = 0$. First we prove a proposition that will later allow us to consider translations of an operator T by a multiple of the identity instead of the operator T itself.

Proposition 2.1. *Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}(\mathcal{X})$ be such that there exists a subspace $Y \subset \mathcal{X}$ for which T is an isomorphism on Y and $d(Y, TY) > 0$. Then for every $\lambda \in \mathbb{C}$, $(T - \lambda I)_{|Y}$ is an isomorphism and $d(Y, (T - \lambda I)Y) > 0$.*

Proof. First, note that the two hypotheses on Y (that T is an isomorphism on Y and $d(Y, TY) > 0$) are together equivalent to the existence of a constant $c > 0$ s.t. for all $y \in S_Y$, $d(Ty, Y) > c$. To see this, let us first assume that the hypotheses of the theorem are satisfied. Then there exists a constant C such that $\|Ty\| \geq C$ for every $y \in S_Y$. For an arbitrary $y \in S_Y$, let $z_y = \frac{Ty}{\|Ty\|}$ and then we clearly have

$$d(Ty, Y) = \|Ty\| d(z_y, Y) \geq C d(TY, Y) =: c > 0.$$

To show the other direction note that for $y \in S_Y$, $0 < c < d(Ty, Y) = \|Ty\| d(z_y, Y) \leq \|T\| d(z_y, Y)$. Taking the infimum over all $z_y \in S_{TY}$ in the last inequality, we obtain that $d(TY, Y) > 0$ and hence $d(Y, TY) > 0$. On the other hand, for all $y \in S_Y$ we have

$$0 < c < d(Ty, Y) \leq \|Ty - \frac{c}{2} y\| \leq \|Ty\| + \frac{c}{2},$$

hence $\|Ty\| \geq \frac{c}{2}$, which in turn implies that T is an isomorphism on Y.

Now it is easy to finish the proof. The condition $d(Ty,Y) > c$ for all $y \in S_Y$ is clearly satisfied if we substitute T with $T-\lambda I$ since for a fixed $y \in S_Y$,

$$d((T-\lambda I)y, Y) = \inf_{z\in Y} \|(T-\lambda I)y - z\| = \inf_{z\in Y} \|Ty - z\| = d(Ty, Y),$$

hence $(T-\lambda I)_{|Y}$ is an isomorphism and $d(Y,(T-\lambda I)Y) > 0$. □

Note the following two simple facts:

– If $T\colon \mathcal{X} \to \mathcal{X}$ is a commutator on $\mathcal{X}$ and $S\colon \mathcal{X} \to \mathcal{Y}$ is an onto isomorphism, then STS^{-1} is a commutator on $\mathcal{Y}$.
– Let $T\colon \mathcal{X} \to \mathcal{X}$ be such that there exists $X_1 \subset \mathcal{X}$ for which $T_{|X_1}$ is an isomorphism and $d(X_1, TX_1) > 0$. If $S\colon \mathcal{X} \to \mathcal{Y}$ is an onto isomorphism, then there exists $Y_1 \subset \mathcal{Y}$, $Y_1 \simeq X_1$, such that $STS^{-1}{}_{|Y_1}$ is an isomorphism and $d(Y_1, STS^{-1}Y_1) > 0$ (in fact $Y_1 = SX_1$). Note also that if X_1 is complemented in $\mathcal{X}$, then Y_1 is complemented in $\mathcal{Y}$.

Using the two facts above, sometimes we will replace an operator T by an operator T_1 which is similar to T and possibly acts on another Banach space.

If $\{Y_i\}_{i=0}^\infty$ is a sequence of arbitrary Banach spaces, by $\left(\sum_{i=0}^\infty Y_i\right)_p$ we denote the space of all sequences $\{y_i\}_{i=0}^\infty$ where $y_i \in Y_i$, $i = 0,1,\ldots$, such that $(\|y_i\|_{Y_i}) \in \ell_p$ with the norm $\|(y_i)\| = \|\|y_i\|_{Y_i}\|_p$ (if $Y_i \equiv Y$ for every $i = 0,1,\ldots$ we will use the notation $\left(\sum Y\right)_p$). We will only consider the case where all the spaces Y_i, $i = 0,1\ldots$, are uniformly isomorphic to a Banach space Y, that is, there exists a constant $\lambda > 0$ and sequence of onto isomorphisms $\{T_i\colon Y_i \to Y\}_{i=0}^\infty$ such that $\|T^{-1}\| = 1$ and $\|T\| \le \lambda$. In this case we define an onto isomorphism $U\colon \left(\sum_{i=0}^\infty Y_i\right)_p \to \left(\sum Y\right)_p$ via (T_i) by

$$U(y_0, y_1, \ldots) = (T_0(y_0), T_1(y_1), \ldots), \tag{2.1}$$

and it is easy to see that $\|U\| \le \lambda$ and $\|U^{-1}\| = 1$. Sometimes we will identify the space $\left(\sum_{i=0}^\infty Y_i\right)_p$ with $(\sum Y)_p$ via the isomorphism U when there is no ambiguity how the properties of an operator T on $\left(\sum_{i=0}^\infty Y_i\right)_p$ translate to the properties of the operator UTU^{-1} on $\left(\sum Y\right)_p$. For $y = (y_i) \in (\sum Y)_p$, $y_i \in Y$, define the following two operators :

$$R(y) = (0, y_0, y_1, \ldots) \qquad , \qquad L(y) = (y_1, y_2, \ldots).$$

The operators L and R are, respectively, the left and the right shift on the space $(\sum Y)_p$. Denote by P_i, $i = 0,1,\ldots$, the natural, norm one, projection from $\left(\sum Y\right)_p$ onto the i-th component of $\left(\sum Y\right)_p$, which we denote by Y^i. We should note that if $Y \simeq \left(\sum Y\right)_p$, then some of the results in this paper are similar to results in [**6**], but initially we do not require this condition, and, in particular, some of the results we prove here have applications to spaces like $\left(\sum \ell_q\right)_p$ for arbitrary $1 \le p, q \le \infty$. Our first proposition shows some basic properties of the left and the right shift as well as the fact that all the powers of L and R are uniformly bounded, which will play an important role in the sequel. Since the proof follows immediately from the definitions we will omit it.

Proposition 2.2. *Consider the Banach space $\left(\sum Y\right)_p$. We have the following identities*

$$\|L^n\| = 1 \ , \ \|R^n\| = 1 \ \text{ for every } n = 1,2,\ldots \tag{2.2}$$

$$LP_0 = P_0R = 0 \ , \ LR = I \ , \ RL = I - P_0 \ , \ RP_i = P_{i+1}R \ , \ P_iL = LP_{i+1} \ \text{ for } i \ge 0. \tag{2.3}$$

Note that we can define a left and right shift on $\left(\sum_{i=0}^\infty Y_i\right)_p$ by $\tilde{L} = U^{-1}LU$ and $\tilde{R} = U^{-1}RU$, and, using the above proposition, we immediately have $\|\tilde{R}^n\| \le \lambda$ and $\|\tilde{L}^n\| \le \lambda$. If there is no ambiguity, we will denote the left and the right shift on $\left(\sum_{i=0}^\infty Y_i\right)_p$ simply by L and R.

Following the ideas in [**3**], for $1 \le p < \infty$ and $p = 0$ define the set

$$\mathcal{A} = \{T \in \mathcal{L}\left(\left(\sum Y\right)_p\right) : \sum_{n=0}^{\infty} R^n T L^n \text{ is strongly convergent}\}, \tag{2.4}$$

and for $T \in \mathcal{A}$ define

$$T_{\mathcal{A}} = \sum_{n=0}^{\infty} R^n T L^n.$$

Now using the fact that an operator T is a commutator if and only if T is in the range of D_S for some S, where D_S is the inner derivation determined by S, defined by $D_S(T) = ST - TS$, it is easy to see ([**6**, Lemma 3]) that if $T \in \mathcal{A}$ then

$$T = D_L(RT_{\mathcal{A}}) = -D_R(T_{\mathcal{A}}L), \tag{2.5}$$

hence T is a commutator.

3. *Commutators on* $\left(\sum Y\right)_p$

The ideas in this section are similar to the ideas in [**6**], but here we present them from a different point of view, in a more general setting and we also include the case $p = \infty$. The following lemma is a generalization of [**3**, Lemma 2.8] in the case $p = \infty$ and [**6**, Corollary 7] in the case $1 \le p < \infty$ and $p = 0$. The proof presented here follows the ideas of the proof in [**6**]. Of course, some of the ideas can be traced back to the classic paper of Brown and Pearcy ([**5**]) and to Apostol's papers [**3**], [**4**], and the references therein.

LEMMA 3.1. *Let* $T \in \mathcal{L}(\left(\sum Y\right)_p)$. *Then the operators* P_0T *and* TP_0 *are commutators.*

Proof. The proof shows that P_0T is in the range of D_L and TP_0 is in the range of D_R. We will consider two cases depending on p.
Case I : $p = \infty$
In this case we first observe that the series

$$S_0 = \sum_{n=0}^{\infty} R^n P_0 T L^n$$

is pointwise convergent coordinatewise. Indeed, let $x \in \left(\sum Y\right)_\infty$ and define $y_n = R^n P_0 T L^n x$ for $n = 0, 1, \ldots$. Note that from the definition we immediately have $y_n \in Y^n$ so the sum $\sum_{n=0}^{\infty} y_n$ converges in the product topology on $\left(\sum Y\right)_\infty$ to a point in $\left(\sum Y\right)_\infty$ since $\|y_n\| \le \|R^n\|\|P_0\|\|T\|\|L^n\|\|x\| \le \|T\|\|x\|$.
Secondly, we observe that S_0 and L commute. Because L and R are continuous operators on $\left(\sum Y\right)_\infty$ with the product topology and $LR = I$, we have

$$\begin{aligned} S_0 L x &= \sum_{n=0}^{\infty} R^n P_0 T L^{n+1} x = L\left(\sum_{n=1}^{\infty} R^n P_0 T L^n x\right) = L\left(\sum_{n=0}^{\infty} R^n P_0 T L^n x\right) - L P_0 T x \\ &= L S_0 x - 0 \end{aligned} \tag{3.1}$$

since $LP_0 = 0$. That is, $D_L S_0 = 0$, as desired.

On the other hand, again using $LP_0 = 0$,

$$(I-RL)S_0x = \sum_{n=0}^{\infty}(I-RL)R^nP_0TL^nx = (I-RL)P_0Tx + \underbrace{\sum_{n=1}^{\infty}(I-RL)R^nP_0TL^nx}_{0} \tag{3.2}$$
$$= (I-RL)P_0Tx = P_0Tx.$$

Therefore

$$D_L(RS_0) = (D_LR)S_0 + R(D_LS_0) = (I-RL)S_0 + 0 = P_0T. \tag{3.3}$$

The proof of the statement that TP_0 is a commutator involves a similar modification of the proof of [**3**, Lemma 2.8]. Again, consider the series

$$S = \sum_{n=0}^{\infty} R^nP_0TP_0L^n.$$

This is pointwise convergent coordinatewise and $SL = LS$ (from the above reasoning applied to the operator TP_0), and

$$D_R(-SL) = -D_R(LS) = -RLS + LSR = -(I-P_0)S + LSR$$
$$= -S + P_0S + SLR = -S + P_0S + S = P_0TP_0.$$

Now it is easy to see that

$$D_R(LTP_0 - SL) = RLTP_0 - \underbrace{LTP_0R}_{0} + P_0TP_0 = (I-P_0)TP_0 + P_0TP_0 = TP_0.$$

Case II : $1 \le p < \infty$ or $p = 0$
In this case the proof is similar to the proof of [**6**, Lemma 6 and Corollary 7] and we include it for completeness. Let us consider the case $p \ge 1$ first. For any $y \in \left(\sum Y\right)_p$ we have

$$\|\sum_{n=m}^{m+r} R^nP_iTP_jL^ny\|^p = \|\sum_{n=m}^{m+r} R^nP_iTP_jL^nP_{j+n}y\|^p = \sum_{n=m}^{m+r}\|R^nP_iTP_jL^nP_{j+n}y\|^p$$
$$\le \|P_iTP_j\|^p\sum_{n=m}^{m+r}\|P_{j+n}y\|^p \le \|P_iTP_j\|^p\sum_{n=m}^{\infty}\|P_{j+n}y\|^p.$$

Since $\sum_{n=m}^{\infty}\|P_{j+n}y\|^p \to 0$ as $m \to \infty$ we have that $\sum_{n=0}^{\infty}R^nP_iTP_jL^n$ is strongly convergent and $P_iTP_j \in \mathcal{A}$.
For $p = 0$ a similar calculation shows

$$\|\sum_{n=m}^{m+r} R^nP_iTP_jL^ny\| = \|\sum_{n=m}^{m+r} R^nP_iTP_jL^nP_{j+n}y\| = \max_{m\le n\le m+r}\|R^nP_iTP_jL^nP_{j+n}y\|$$
$$\le \|P_iTP_j\|\max_{m\le n\le m+r}\|P_{j+n}y\|$$

and since $\max_{m\le n\le m+r}\|P_{j+n}y\| \to 0$ as $m \to \infty$ we apply the same argument as in the case $p \ge 1$ to obtain $P_iTP_j \in \mathcal{A}$.
Using $P_iTP_j \in \mathcal{A}$ for $i = j = 0$ and (2.5) we have $P_0TP_0 = D_L(R(P_0TP_0)_{\mathcal{A}}) = -D_R((P_0TP_0)_{\mathcal{A}}L)$. Again, as in [**6**, Corollary 7], via direct computation we obtain

$$TP_0 = D_R(LTP_0 - (P_0TP_0)_{\mathcal{A}}L) \tag{3.4}$$
$$P_0T = D_L(-P_0TR + R(P_0TP_0)_{\mathcal{A}}). \tag{3.5}$$

□

Now we switch our attention to Banach spaces which in addition satisfy $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$ for some $1 \le p \le \infty$ or $p = 0$. Note that the Banach space $\left(\sum Y\right)_p$ satisfies this condition regardless of the space Y, hence we will be able to use the results we proved so far in this section. We begin with a definition.

DEFINITION 1. Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$, $1 \le p \le \infty$ or $p = 0$. We say that $\mathcal{D} = \{X_i\}_{i=0}^{\infty}$ is a decomposition of $\mathcal{X}$ if it forms an ℓ_p or c_0 decomposition of $\mathcal{X}$ into subspaces which are uniformly isomorphic to $\mathcal{X}$; that is, if the following three conditions are satisfied:

- There are uniformly bounded projections P_i on $\mathcal{X}$ with $P_i\mathcal{X} = X_i$ and $P_iP_j = 0$ for $i, j = 0, 1, \dots$ and $i \neq j$
- There exists a collection of isomorphisms $\psi_i : X_i \to \mathcal{X}$, $i \in \mathbb{N}_0$ (we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), such that $\|\psi_i^{-1}\| = 1$ and $\lambda = \sup\limits_{i \in \mathbb{N}_0} \|\psi_i\| < \infty$
- The formula $Sx = (\psi_i P_i x)$ defines a surjective isomorphism from $\mathcal{X}$ onto $\left(\sum \mathcal{X}\right)_p$

If $\mathcal{D} = \{X_i\}_{i=0}^{\infty}$ is a decomposition of $\mathcal{X}$ we have $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p \simeq (\sum_{i=0}^{\infty} X_i)_p$, where the second isomorphic relation is via the isomorphism U defined in (2.1). Using this simple observation we will often identify $\mathcal{X}$ with $(\sum_{i=0}^{\infty} X_i)_p$. Our next theorem is similar to [**6**, Theorem 16] and [**3**, Theorem 4.6], but we state it and prove it in a more general setting and also include the case $p = \infty$.

THEOREM 3.2. *Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$, $1 \le p \le \infty$ or $p = 0$. Let $T \in \mathcal{L}(\mathcal{X})$ be such that there exists a subspace $X \subset \mathcal{X}$ such that $X \simeq \mathcal{X}$, $T_{|X}$ is an isomorphism, $X + T(X)$ is complemented in $\mathcal{X}$ and $d(X, T(X)) > 0$. Then there exists a decomposition $\mathcal{D}$ of $\mathcal{X}$ such that T is similar to a matrix operator of the form*

$$\begin{pmatrix} * & L \\ * & * \end{pmatrix}$$

on $\mathcal{X} \oplus \mathcal{X}$, where L is the left shift associated with $\mathcal{D}$.

Proof. Clearly $\mathcal{X} = X \oplus T(X) \oplus Z$ where Z is complemented in $\mathcal{X}$. Note that without loss of generality we can assume that Z is isomorphic to $\mathcal{X}$. Indeed, if this is not the case, let $X = X_1 \oplus X_2$, $X \simeq X_1 \simeq X_2$ and X_1, X_2 complemented in X (hence also complemented in $\mathcal{X}$). Then $d(X_1, T(X_1)) > 0$ and $\mathcal{X} = X_1 \oplus T(X_1) \oplus Z_1$ where Z_1 is a complemented subspace of $\mathcal{X}$, which contains the subspace $X_2 \subset \mathcal{X}$, such that X_2 is isomorphic to $\mathcal{X}$ and complemented in Z. Applying the Pełczýnski decomposition technique ([**14**, Proposition 4]), we conclude that Z_1 is isomorphic to X. This observation plays an important role and will allow us to construct the decompositions we need during the rest of the proof.
Denote by $I - P$ the projection onto $T(X)$ with kernel $X + Z$. Consider two decompositions $\mathcal{D}_1 = \{X_i\}_{i=0}^{\infty}$, $\mathcal{D}_2 = \{Y_i\}_{i=0}^{\infty}$ of $\mathcal{X}$ such that $T(X) = Y_0 = X_1 \oplus X_2 \oplus \cdots$, $X_0 = Y_1 \oplus Y_2 \oplus \cdots$, $Y_1 = X$, and $Z = Y_2 \oplus Y_3 \oplus \cdots$. Define a map S

$$S\varphi = L_{\mathcal{D}_1}\varphi \oplus L_{\mathcal{D}_2}\varphi, \qquad \varphi \in \mathcal{X}$$

from $\mathcal{X}$ to $\mathcal{X} \oplus \mathcal{X}$. The map S is invertible ($S^{-1}(a, b) = R_{\mathcal{D}_1}a + R_{\mathcal{D}_2}b$). Just using the definition of S and the formula for S^{-1} we see that

$$\begin{aligned} STS^{-1}(a,b) &= ST(R_{\mathcal{D}_1}a + R_{\mathcal{D}_2}b) = S(TR_{\mathcal{D}_1}a + TR_{\mathcal{D}_2}b) \\ &= (L_{\mathcal{D}_1}TR_{\mathcal{D}_1}a + L_{\mathcal{D}_1}TR_{\mathcal{D}_2}b) \oplus (L_{\mathcal{D}_2}TR_{\mathcal{D}_1}a + L_{\mathcal{D}_2}TR_{\mathcal{D}_2}b), \end{aligned}$$

hence

$$STS^{-1} = \begin{pmatrix} * & L_{\mathcal{D}_1}TR_{\mathcal{D}_2} \\ * & * \end{pmatrix}.$$

Let

$$A = P_{Y_0}TR_{\mathcal{D}_2} = (I-P)TR_{\mathcal{D}_2} \tag{3.6}$$

and note that $A_{|P_{Y_0}\mathcal{X}} \equiv A_{|(I-P)\mathcal{X}} : (I-P)\mathcal{X} \to (I-P)\mathcal{X}$ is onto and invertible since $R_{\mathcal{D}_2}$ is an isomorphism on $P_{Y_0}\mathcal{X}$ and $R_{\mathcal{D}_2}(P_{Y_0}\mathcal{X}) = Y_1 = X$. Here we used the fact that $P_{Y_0}T$ is an isomorphism on X ($PX = X$). Denote by $T_0 : (I-P)\mathcal{X} \to (I-P)\mathcal{X}$ the inverse of $A_{|P_{Y_0}\mathcal{X}}$ (note that T_0 is an automorphism on $(I-P)\mathcal{X}$) and consider $G : \mathcal{X} \to \mathcal{X}$ defined by

$$G = I + T_0(I-P) - T_0A.$$

We will show that $G^{-1} = A + P$. In fact, from the definitions of A and T_0 it is clear that

$$AT_0(I-P) = T_0A(I-P) = I-P\,,\ PT_0(I-P) = PA = 0\,,\ (I-P)A = A \tag{3.7}$$

and since A maps onto $(I-P)\mathcal{X}$ and $AT_0 = I_{|(I-P)\mathcal{X}}$ we also have

$$A - AT_0A = 0. \tag{3.8}$$

Now using (3.7) and (3.8) we compute

$$\begin{aligned} (A+P)G &= (A+P)(I + T_0(I-P) - T_0A) \\ &= A + AT_0(I-P) - AT_0A + P = I - P + P = I \\ G(A+P) &= (I + T_0(I-P) - T_0A)(A+P) \\ &= A + P + T_0(I-P)A + T_0(I-P)P - T_0AA - T_0AP \\ &= A + P + T_0A - T_0AA - T_0AP \\ &= P + (I - T_0A)A + T_0A(I-P) \\ &= P + (I - T_0A)(I-P)A + (I-P) \\ &= I + ((I-P) - T_0A(I-P))A \\ &= I + (I - P - (I-P))A = I. \end{aligned}$$

Using a similarity we obtain

$$\begin{pmatrix} I & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} * & L_{\mathcal{D}_1}TR_{\mathcal{D}_2} \\ * & * \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} = \begin{pmatrix} * & L_{\mathcal{D}_1}TR_{\mathcal{D}_2}G \\ * & * \end{pmatrix}.$$

It is clear that we will be done if we show that $L_{\mathcal{D}_1} = L_{\mathcal{D}_1}TR_{\mathcal{D}_2}G$. In order to do this consider the equation $(A+P)G = I \Leftrightarrow AG + PG = I$. Multiplying both sides of the last equation on the left by $L_{\mathcal{D}_1}$ gives us $L_{\mathcal{D}_1}AG + L_{\mathcal{D}_1}PG = L_{\mathcal{D}_1}$. Using $L_{\mathcal{D}_1}P \equiv L_{\mathcal{D}_1}P_{X_0} = 0$ we obtain $L_{\mathcal{D}_1}AG = L_{\mathcal{D}_1}$. Finally, substituting A from (3.6) in the last equation yields

$$L_{\mathcal{D}_1} = L_{\mathcal{D}_1}AG = L_{\mathcal{D}_1}P_{Y_0}TR_{\mathcal{D}_2}G = L_{\mathcal{D}_1}(I - P_{X_0})TR_{\mathcal{D}_2}G = L_{\mathcal{D}_1}TR_{\mathcal{D}_2}G,$$

which finishes the proof. □

The following theorem was proved in [**3**] for $X = \ell_p$, $1 < p < \infty$, but inessential modifications give the result in these general settings.

THEOREM 3.3. *Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$. Let $\mathcal{D}$ be a decomposition of $\mathcal{X}$ and let L be the left shift associated with it. Then the matrix operator*

$$\begin{pmatrix} T_1 & L \\ T_2 & T_3 \end{pmatrix}$$

acting on $\mathcal{X} \oplus \mathcal{X}$ is a commutator.

Proof. Let $\mathcal{D} = \{X_i\}$ be the given decomposition. Consider a decomposition $\mathcal{D}_1 = \{Y_i\}$ such that $Y_0 = \bigoplus_{i=1}^{\infty} X_i$ and $X_0 = \bigoplus_{i=1}^{\infty} Y_i$. Now there exists an operator G such that $D_{L_\mathcal{D}} G = R_{\mathcal{D}_1} L_{\mathcal{D}_1}(T_1 + T_3)$. This can be done using Lemma 3.1, since $R_{\mathcal{D}_1} L_{\mathcal{D}_1} = I - P_{Y_0} = P_{X_0}$. Note that we have $T_1 + T_3 - LG + GL = T_1 + T_3 - D_L G = T_1 + T_3 - R_{\mathcal{D}_1} L_{\mathcal{D}_1}(T_1 + T_3) = P_{Y_0}(T_1 + T_3)$, and using Lemma 3.1 again, we deduce that $T_1 + T_3 - LG + GL$ is a commutator. Thus by making the similarity

$$\widetilde{T} := \begin{pmatrix} I & 0 \\ G & I \end{pmatrix} \begin{pmatrix} T_1 & L \\ T_2 & T_3 \end{pmatrix} \begin{pmatrix} I & 0 \\ -G & I \end{pmatrix} = \begin{pmatrix} T_1 - LG & L \\ * & T_3 + GL \end{pmatrix}$$

and replacing T by $\widetilde{T}$ we can assume that $T_1 + T_3$ is a commutator, say $T_1 + T_3 = AB - BA$ and $\|A\| < 1/(2\|R\|)$ (this can be done by scaling). Denote by M_S left multiplication by an operator S. Then $\|M_R D_A\| < 1$ where R is the right shift associated with $\mathcal{D}$. The operator $T_0 = (M_I - M_R D_A)^{-1} M_R(T_3 B - T_2)$ is well defined and it is easy to see that

$$\begin{pmatrix} A & 0 \\ T_3 & A - L \end{pmatrix} \begin{pmatrix} B & I \\ T_0 & 0 \end{pmatrix} - \begin{pmatrix} B & I \\ T_0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ T_3 & A - L \end{pmatrix} = \begin{pmatrix} T_1 & L \\ T_2 & T_3 \end{pmatrix}.$$

This finishes the proof. □

4. *Operators on ℓ_∞*

DEFINITION 2. The left essential spectrum of $T \in \mathcal{L}(\mathcal{X})$ is the set ([**2**] Def 1.1)

$$\sigma_{l.e.}(T) = \{\lambda \in \mathbb{C} : \inf_{x \in S_Y} \|(\lambda - T)x\| = 0 \text{ for all } Y \subset \mathcal{X} \text{ s.t. } \operatorname{codim}(Y) < \infty\}.$$

Apostol [**2**, Theorem 1.4] proved that for any $T \in \mathcal{L}(\mathcal{X})$, $\sigma_{l.e.}(T)$ is a closed non-void set. The following lemma is a characterization of the operators not of the form $\lambda I + K$ on the classical Banach sequence spaces. The proof presented here follows Apostol's ideas [**3**, Lemma 4.1], but it is presented in a more general way.

LEMMA 4.1. *Let $\mathcal{X}$ be a Banach space isomorphic to ℓ_p for $1 \le p < \infty$ or c_0 and let $T \in \mathcal{L}(\mathcal{X})$. Then the following are equivalent*
(1) *$T - \lambda I$ is not a compact operator for any $\lambda \in \mathbb{C}$.*
(2) *There exists an infinite dimensional complemented subspace $Y \subset \mathcal{X}$ such that $Y \simeq \mathcal{X}$, $T_{|Y}$ is an isomorphism and $d(Y, T(Y)) > 0$.*

Proof. **(2) $\Longrightarrow$ (1)**
Assume that $T = \lambda I + K$ for some $\lambda \in \mathbb{C}$ and some $K \in \mathcal{K}(\mathcal{X})$. Clearly $\lambda \neq 0$ since $T_{|Y}$ is an isomorphism. Now there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset S_Y$ such that $\|Kx_n\| \to 0$ as $n \to \infty$. Let

$y_n = T\left(\frac{x_n}{\lambda}\right)$ and note that

$$\|x_n - y_n\| = \left\|x_n - (\lambda I + K)\left(\frac{x_n}{\lambda}\right)\right\| = \left\|x_n - x_n - K\left(\frac{x_n}{\lambda}\right)\right\| = \frac{\|Kx_n\|}{\lambda} \to 0$$

as $n \to \infty$ which contradicts the assumption $d(Y, T(Y)) > 0$. Thus $T - \lambda I$ is not a compact operator for any $\lambda \in \mathbb{C}$.

(1) $\Longrightarrow$ (2)

The proof in this direction follows the ideas of the proof of Lemma 4.1 from [**3**]. Let $\lambda \in \sigma_{l.e.}(T)$. Then $T_1 = T - \lambda I$ is not a compact operator and $0 \in \sigma_{l.e}(T_1)$. Using just the definition of the left essential spectrum, we find a normalized block basis sequence $\{x_i\}_{i=1}^\infty$ of the standard unit vector basis of $\mathcal{X}$ such that $\|T_1 x_n\| < \frac{1}{2^n}$ for $n = 1, 2, \ldots$. Thus if we denote $Z = \overline{\text{span}}\{x_i : i = 1, 2, \ldots\}$ we have $Z \simeq \mathcal{X}$ and $T_{1|Z}$ is a compact operator. Let $I - P$ be a bounded projection from $\mathcal{X}$ onto Z ([**14**, Lemma 1]) so that $T_1(I - P)$ is compact. Now consider the operator $T_2 = (I - P)T_1 P$. We have two possibilities:

Case I. Assume that $T_2 = (I - P)T_1 P$ is not a compact operator. Then there exists an infinite dimensional subspace $Y_1 \subset P\mathcal{X}$ on which T_2 is an isomorphism and hence using [**14**, Lemma 2] if necessary, we find a complemented subspace $Y \subset P\mathcal{X}$, such that T_2 is an isomorphism on Y. By the construction of the operator T_2 we immediately have $d(Y, (I - P)T_1 P(Y)) > 0$ and hence $d(Y, T_1(Y)) > 0$. Note that since $\mathcal{X}$ is prime and Y is complemented in $\mathcal{X}$, $Y \simeq \mathcal{X}$ is automatic. Now we are in position to use Proposition 2.1 to conclude that $d(Y, T(Y)) > 0$.

Case II. Now we can assume that the operator $(I - P)T_1 P$ is compact. Since $T_1(I - P)$ is compact and using

$$T_1 = T_1(I - P) + (I - P)T_1 P + PT_1 P$$

we conclude that the operator PT_1P is not compact. Using $\mathcal{X} \equiv P\mathcal{X} \oplus (I - P)\mathcal{X}$, we identify $P\mathcal{X} \oplus (I - P)\mathcal{X}$ with $\mathcal{X} \oplus \mathcal{X}$ via an onto isomorphism U, such that U maps $P\mathcal{X}$ onto the first copy of $\mathcal{X}$ in the sum $\mathcal{X} \oplus \mathcal{X}$. Without loss of generality we assume that $T_1 = \left(\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right)$ is acting on $\mathcal{X} \oplus \mathcal{X}$. Denote by $P = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ the projection from $\mathcal{X} \oplus \mathcal{X}$ onto the first copy of $\mathcal{X}$. In the new settings, we have that T_{11} is not compact and T_{21}, T_{22} and T_{12} are compact operators. Define the operator S on $\mathcal{X} \oplus \mathcal{X}$ in the following way:

$$\sqrt{2}S = \begin{pmatrix} I & I \\ I & -I \end{pmatrix}.$$

Clearly $S^2 = I$ hence $S = S^{-1}$. Now consider the operator $2(I - P)S^{-1}T_1 SP$. A simple calculation shows that

$$2(I - P)S^{-1}T_1 SP = \begin{pmatrix} 0 & 0 \\ T_{11} + T_{12} - T_{21} - T_{22} & 0 \end{pmatrix}$$

hence $(I - P)S^{-1}T_1 SP$ is not compact. Now we can continue as in the previous case to conclude that there exists a complemented subspace $Y \subset \mathcal{X}$ in the first copy of $\mathcal{X} \oplus \mathcal{X}$ for which $d(Y, S^{-1}T_1 S(Y)) > 0$ and hence $d(SY, T_1(SY)) > 0$. Again using Proposition 2.1, we conclude that $d(SY, T(SY)) > 0$. □

Remark 1. We should note that the two conditions in the preceding lemma are equivalent to a third one, which is the same as **(2)** plus the additional condition that $Y \oplus T(Y)$ is complemented in $\mathcal{X}$. This is essentially what was used for proving the complete classification of the commutators on ℓ_1 in [**6**], and ℓ_p, $1 < p < \infty$, and c_0 in [**3**] and [**4**]. The last mentioned condition will also play an important role in the proof of the complete classification of the commutators on ℓ_∞, but we should point out that once we have an infinite dimensional subspace $Y \subset \ell_\infty$ such that $Y \simeq \ell_\infty$, $T_{|Y}$ is an isomorphism and $d(Y, T(Y)) > 0$, then Y and $Y \oplus T(Y)$ will be automatically complemented in ℓ_∞.

Lemma 4.2. *Let $T \in \mathcal{L}(\ell_\infty)$ and denote by I the identity operator on ℓ_∞. Then the following are equivalent*
(a) For each subspace $X \subset \ell_\infty, X \simeq c_0$, there exists a constant λ_X and a compact operator $K_X : X \to \ell_\infty$ depending on X such that $T_{|X} = \lambda_X I_{|X} + K_X$.
(b) There exists a constant λ such that $T = \lambda I + S$, where $S \in \mathcal{S}(\ell_\infty)$.

Proof. Clearly (b) implies (a), since every strictly singular operator from c_0 to any Banach space is compact ([**1**, Theorem 2.4.10]). For proving the other direction we will first show that for every two subspaces X, Y such that $X \simeq Y \simeq c_0$ we have $\lambda_X = \lambda_Y$. We have several cases.

Case I. $X \cap Y = \{0\},\ d(X,Y) > 0$.
Let $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ be bases for X and Y, respectively, which are equivalent to the usual unit vector basis of c_0. Consider the sequence $\{z_i\}_{i=1}^\infty$ such that $z_{2i} = x_i$, $z_{2i-1} = y_i$ for $i = 1, 2, \ldots$. If we denote $Z = \overline{\text{span}}\{z_i : i = 1, 2, \ldots\}$, then clearly $Z \simeq c_0$, and, using the assumption of the lemma, we have that $T_{|Z} = \lambda_Z I_{|Z} + K_Z$. Now using $X \subset Z$ we have that $\lambda_X I_{|X} + K_X = (\lambda_Z I_{|Z} + K_Z)_{|X}$, hence

$$(\lambda_X - \lambda_Z) I_{|X} = (K_Z)_{|X} - K_X.$$

The last equation is only possible if $\lambda_X = \lambda_Z$ since the identity is never a compact operator on a infinite dimensional subspace. Similarly $\lambda_Y = \lambda_Z$ and hence $\lambda_X = \lambda_Y$.

Case II. $X \cap Y = \{0\},\ d(X,Y) = 0$.
Again let $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ be bases of X and Y, respectively, which are equivalent to the usual unit vector basis of c_0 and assume also that $\lambda_X \neq \lambda_Y$. There exists a normalized block basis $\{u_i\}_{i=1}^\infty$ of $\{x_i\}_{i=1}^\infty$ and a normalized block basis $\{v_i\}_{i=1}^\infty$ of $\{y_i\}_{i=1}^\infty$ such that $\|u_i - v_i\| < \dfrac{1}{i}$. Then $\|u_i - v_i\| \to 0 \Rightarrow \|Tu_i - Tv_i\| \to 0 \Rightarrow \|\lambda_X u_i + K_X u_i - \lambda_Y v_i - K_Y v_i\| \to 0$. Since $u_i \to 0$ weakly (as a bounded block basis of the standard unit vector basis of c_0) we have $\|K_X u_i\| \to 0$ and using $\|u_i - v_i\| \to 0$ we conclude that

$$\|(\lambda_X - \lambda_Y) v_i - K_Y v_i\| \to 0.$$

Then there exists $N \in \mathbb{N}$ such that $\|K_Y v_i\| > \frac{|\lambda_X - \lambda_Y|}{2} \|v_i\|$ for $i > N$, which is impossible because K_Y is a compact operator. Thus, in this case we also have $\lambda_X = \lambda_Y$.

Case III. $X \cap Y = Z \neq \{0\},\ \dim(Z) = \infty$.
In this case we have $(\lambda_X I_{|X} + K_X)_{|Z} = (\lambda_Y I_{|Y} + K_Y)_{|Z}$ and, as in the first case, we rewrite the preceding equation in the form

$$(\lambda_X I_{|X} - \lambda_Y I_{|Y})_{|Z} = (K_Y - K_X)_{|Z}.$$

Again, as in **Case I**, the last equation is only possible if $\lambda_X = \lambda_Y$ since the identity is never a compact operator on a infinite dimensional subspace.

Case IV. $X \cap Y = Z \neq \{0\},\ \dim(Z) < \infty$.
Let $X = Z \bigoplus X_1$ and $Y = Z \bigoplus Y_1$. Then $X_1 \cap Y_1 = \{0\}$, $X_1 \simeq Y_1 \simeq c_0$ and we can reduce to one of the previous cases.

Let us denote $S = T - \lambda I$ where $\lambda = \lambda_X$ for arbitrary $X \subset \ell_\infty$, $X \simeq c_0$. If S is not a strictly singular operator, then there is a subspace $Z \subset \ell_\infty$, $Z \simeq \ell_\infty$ such that $S_{|Z}$ is an isomorphism ([**16**, Corollary 1.4]), hence we can find $Z_1 \subset Z \subset \ell_\infty$, $Z_1 \simeq c_0$, such that $S_{|Z_1}$ is an isomorphism. This contradicts the assumption that $S_{|Z_1}$ is a compact operator. □

The following corollary is an immediate consequence of Lemma 4.2.

Corollary 4.3. *Suppose $T \in \mathcal{L}(\ell_\infty)$ is such that $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any $\lambda \in \mathbb{C}$. Then there exists a subspace $X \subset \ell_\infty$, $X \simeq c_0$ such that $(T-\lambda I)_{|X}$ is not a compact operator for any $\lambda \in \mathbb{C}$.*

Theorem 4.4. *Let $T \in \mathcal{L}(\ell_\infty)$ be such that $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any λ. Then there exists a subspace $X \subset \ell_\infty$ such that $X \simeq c_0$, $T_{|X}$ is an isomorphism and $d(X, T(X)) > 0$.*

Proof. By Corollary 4.3 we have a subspace $X \subset \ell_\infty$, $X \simeq c_0$ such that $(T-\lambda I)_{|X}$ is not a compact operator for any λ. Let $Z = \overline{X \oplus T(X)}$ and let P be a projection from Z onto X (such exists since Z is separable and $X \simeq c_0$). We have two cases:
Case I. The operator $T_1 = (I-P)TP$ is not compact. Since T_1 is a non-compact operator from $X \simeq c_0$ into a Banach space we have that T_1 is an isomorphism on some subspace $Y \subset X$, $Y \simeq c_0$ ([**1**, Theorem 2.4.10]). Clearly, from the form of the operator T_1 we have $d(Y, T_1(Y)) = d(Y, (I-P)TP(Y)) > 0$ and hence $d(Y, T(Y)) > 0$.
Case II. If $(I-P)TP$ is compact and $\lambda \in \mathbb{C}$, then $(I-P)TP + PTP - \lambda I_{|Z} = TP - \lambda I_{|Z}$ is not compact and hence $PTP - \lambda I_{|Z}$ is not compact. Now for $T_2 := PTP\colon X \to X$ we apply Lemma 4.1 to conclude that there exists a subspace $Y \subseteq X$, $Y \simeq c_0$ such that $d(Y, PT(Y)) = d(Y, PTP(Y)) > 0$ and hence $d(Y, T(Y)) > 0$. □

The following theorem is an analog of Lemma 4.1 for the space ℓ_∞.

Theorem 4.5. *Let $T \in \mathcal{L}(\ell_\infty)$ be such that $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any $\lambda \in \mathbb{C}$. Then there exists a subspace $X \subset \ell_\infty$ such that $X \simeq \ell_\infty$, $T_{|X}$ is an isomorphism and $d(X, T(X)) > 0$.*

Proof. From Theorem 4.4 we have a subspace $Y \subset \ell_\infty$, $Y \simeq c_0$ such that $T_{|Y}$ is an isomorphism and $d(Y, T(Y)) > 0$. Let $N_k = \{3i + k \colon i = 0, 1, \dots\}$ for $k = 1, 2, 3$. There exists an isomorphism $\overline{S}\colon Y \oplus TY \to c_0(N_1) \oplus c_0(N_2)$ such that $\overline{S}(Y) = c_0(N_1)$ and $\overline{S}(TY) = c_0(N_2)$. Note that the space $Y \oplus TY$ is indeed a closed subspace of ℓ_∞ due to the fact that $d(Y, T(Y)) > 0$. Now we use [**12**, Theorem 3] to extend $\overline{S}$ to an automorphism S on ℓ_∞. Let $T_1 = STS^{-1}$ and consider the operator $(P_{N_2}T_1)_{|\ell_\infty(N_1)}\colon \ell_\infty(N_1) \to \ell_\infty(N_2)$, where P_{N_2} is the natural projection onto $\ell_\infty(N_2)$. Since $T_1(c_0(N_1)) = c_0(N_2)$, by [**16**, Proposition 1.2] there exists an infinite set $M \subset N_1$ such that $(P_{N_2}T_1)_{|\ell_\infty(M)}$ is an isomorphism. This immediately yields

$$d(\ell_\infty(M), P_{N_2}T_1(\ell_\infty(M))) > 0.$$

If $x \in \ell_\infty(M)$, $\|x\| = 1$ and $y \in \ell_\infty(M)$ is arbitrary, then

$$\|x - Ty\| = \max(\|x - P_M T_1 y\|, \|P_{M^c} T_1 y\|) \geq \max(\|x - P_M T_1 y\|, \|P_{N_2} T_1 y\|)$$

If $\|y\| < \frac{1}{2\|T_1\|}$ then $\|x - P_M T_1 y\| \geq \frac{1}{2}$. Otherwise $\|P_{N_2} T_1 y\| \geq \frac{1}{2\|T\|\|(P_{N_2}T_1)^{-1}\|}$ where the norm of the inverse of $P_{N_2}T_1$ in the preceding inequality is taken considering $P_{N_2}T_1$ as an operator from $\ell_\infty(M)$ to $P_{N_2}T_1(\ell_\infty(M))$. Now it is clear that

$$\|x - Ty\| \geq \max\left(\frac{1}{2}, \frac{1}{2\|T\|\|(P_{N_2}T_1)^{-1}\|}\right)$$

for all $x \in \ell_\infty(M)$, $\|x\| = 1$ and $y \in \ell_\infty(M)$ hence

$$d(\ell_\infty(M), T_1(\ell_\infty(M))) > 0. \tag{4.1}$$

Finally, recall that $T_1 = STS^{-1}$, thus

$$d(\ell_\infty(M), STS^{-1}(\ell_\infty(M))) > 0$$

and hence $d(S^{-1}(\ell_\infty(M)), TS^{-1}(\ell_\infty(M))) > 0$. □

Finally, we can prove our main result.

THEOREM 4.6. *An operator $T \in \mathcal{L}(\ell_\infty)$ is a commutator if and only if $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any $\lambda \neq 0$.*

Proof. Note first that if T is a commutator, from the remarks we made in the introduction it follows that $T - \lambda I$ cannot be strictly singular for any $\lambda \neq 0$. For proving the other direction we have to consider two cases:
Case I. If $T \in \mathcal{S}(\ell_\infty)$ $(\lambda = 0)$, the statement of the theorem follows from [**6**, Theorem 23].
Case II. If $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any $\lambda \in \mathbb{C}$, then we apply Theorem 4.5 to get $X \subset \ell_\infty$ such that $X \simeq \ell_\infty$, $T_{|X}$ an isomorphism and $d(X, TX) > 0$. The subspace $X + TX$ is isomorphic to ℓ_∞ and thus is complemented in ℓ_∞. Theorem 3.2 now yields that T is similar to an operator of the form $\begin{pmatrix} * & L \\ * & * \end{pmatrix}$. Finally, we apply Theorem 3.3 to complete the proof. □

5. *Remarks and problems*

We end this note with some comments and questions that arise from our work.
First consider the set

$$\mathcal{M}_{\mathcal{X}} = \{T \in \mathcal{L}(\mathcal{X}) \,:\, I_{\mathcal{X}} \text{ does not factor through } T\}.$$

This set comes naturally from our investigation of the commutators on ℓ_p for $1 \leq p \leq \infty$. We know ([**6**, Theorem 18], [**3**, Theorem 4.8], [**4**, Theorem 2.6]) that the non-commutators on ℓ_p, $1 \leq p < \infty$ and c_0 have the form $\lambda I + K$ where $K \in \mathcal{M}_{\mathcal{X}}$ and $\lambda \neq 0$, where $\mathcal{M}_{\mathcal{X}} = \mathcal{K}(\ell_p)$ is actually the largest ideal in $\mathcal{L}(\ell_p)$ ([**8**]), and, in this paper we showed (Theorem 4.6) that the non-commutators on ℓ_∞ have the form $\lambda I + S$ where $S \in \mathcal{M}_{\mathcal{X}}$ and $\lambda \neq 0$, where $\mathcal{M}_{\mathcal{X}} = \mathcal{S}(\ell_\infty)$. Thus, it is natural to ask the question for which Banach spaces $\mathcal{X}$ is the set $\mathcal{M}_{\mathcal{X}}$ the largest ideal in $\mathcal{L}(\mathcal{X})$? Let us also mention that in addition to the already mentioned spaces, if $\mathcal{X} = L_p(0,1)$, $1 \leq p < \infty$, then $\mathcal{M}_{\mathcal{X}}$ is again the largest ideal in $\mathcal{L}(\mathcal{X})$ (cf. [**7**] for the case $p = 1$ and [**9**, Proposition 9.11] for $p > 1$).

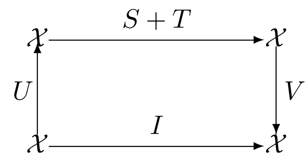
First note that the set $\mathcal{M}_{\mathcal{X}}$ is closed under left and right multiplication with operators from $\mathcal{L}(\mathcal{X})$, so the question whether $\mathcal{M}_{\mathcal{X}}$ is an ideal is equivalent to the question whether $\mathcal{M}_{\mathcal{X}}$ is closed under addition. Note also that if $\mathcal{M}_{\mathcal{X}}$ is an ideal then it is automatically the largest ideal in $\mathcal{L}(\mathcal{X})$ and hence closed, so the question we will consider is under what conditions we have

$$\mathcal{M}_{\mathcal{X}} + \mathcal{M}_{\mathcal{X}} \subseteq \mathcal{M}_{\mathcal{X}}. \tag{5.1}$$

The following proposition gives a sufficient condition for (5.1) to hold.

PROPOSITION 5.1. *Let $\mathcal{X}$ be a Banach space such that for every $T \in \mathcal{L}(\mathcal{X})$ we have $T \notin \mathcal{M}_{\mathcal{X}}$ or $I - T \notin \mathcal{M}_{\mathcal{X}}$. Then $\mathcal{M}_{\mathcal{X}}$ is the largest (hence closed) ideal in $\mathcal{L}(\mathcal{X})$.*

Proof. Let $S, T \in \mathcal{M}_{\mathcal{X}}$ and assume that $S+T \notin \mathcal{M}_{\mathcal{X}}$. By our assumption, there exist two operators $U\colon \mathcal{X} \to \mathcal{X}$ and $V\colon \mathcal{X} \to \mathcal{X}$ which make the following diagram commute:



Denote $W = (S+T)U(\mathcal{X})$ and let $P\colon \mathcal{X} \to W$ be a projection onto W (we can take $P = (S+T)UV$). Clearly $VP(S+T)U = I$. Now $S, T \in \mathcal{M}_{\mathcal{X}}$ implies $VPSU, VPST \in \mathcal{M}_{\mathcal{X}}$ which is a contradiction since $VPSU + VPTU = I$. □

Let us just mention that the conditions of the proposition above are satisfied for $\mathcal{X} = C([0,1])$ ([**11**, Proposition 2.1]) hence $\mathcal{M}_{\mathcal{X}}$ is the largest ideal in $\mathcal{L}(C([0,1]))$ as well.

We should point out that there are Banach spaces for which $\mathcal{M}_{\mathcal{X}}$ is not an ideal in $\mathcal{L}(\mathcal{X})$. In the space $\ell_p \oplus \ell_q$, $1 \le p < q < \infty$, there are exactly two maximal ideals ([**15**]), namely, the closure of the ideal of the operators that factor through ℓ_p, which we will denote by α_p, and the closure of the ideal of the operators that factor through ℓ_q, which we will denote by α_q. In this particular space, the first author proved a necessary and sufficient condition for an operator to be a commutator:

THEOREM 5.2. *([**6**, Theorem 20]) Let P_{ℓ_p} and P_{ℓ_q} be the natural projections from $\ell_p \oplus \ell_q$ onto ℓ_p and ℓ_q, respectively. Then T is a commutator if and only of $P_{\ell_p} T P_{\ell_p}$ and $P_{\ell_q} T P_{\ell_q}$ are commutators as operators acting on ℓ_p and ℓ_q respectively.*

If we denote $T = \left(\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right)$, the last theorem implies that T is not a commutator if and only if T_{11} or T_{22} is not a commutator as an operator acting on ℓ_p or ℓ_q respectively. Now using the classification of the commutators on ℓ_p for $1 \le p < \infty$ and the results in [**15**], it is easy to deduce that an operator on $\ell_p \oplus \ell_q$ is not a commutator if and only if it has the form $\lambda I + K$ where $\lambda \neq 0$ and $K \in \alpha_p \cup \alpha_q$. We can generalize this fact, but first we need a definition and a lemma that follows easily from [**6**, Corollary 21].

Property P. We say that a Banach space $\mathcal{X}$ has property **P** if $T \in \mathcal{L}(\mathcal{X})$ is not a commutator if and only if $T = \lambda I + S$, where $\lambda \neq 0$ and S belongs to some proper ideal of $\mathcal{L}(\mathcal{X})$.

All the Banach spaces we have considered so far have property **P** and our goal now is to show that property **P** is closed under taking finite sums under certain conditions imposed on the elements of the sum.

LEMMA 5.3. *Let $\{X_i\}_{i=1}^n$ be a finite sequence of Banach spaces that have property **P**. Assume also that all operators $A\colon X_i \to X_i$ that factor through X_j are in the intersection of all maximal ideals in $\mathcal{L}(X_i)$ for each $i, j = 1, 2, \ldots, n,\ i \neq j$. Let $\mathcal{X} = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ and let P_i be the natural projections from $\mathcal{X}$ onto X_i for $i = 1, 2, \ldots, n$. Then $T \in \mathcal{L}(\mathcal{X})$ is a commutator if and only if for each $1 \le i \le n$, $P_i T P_i$ is a commutator as an operator acting on X_i.*

Proof. The proof is by induction and it mimics the proof of [**6**, Corollary 21]. First consider the case $n = 2$. Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A : X_1 \to X_1, D : X_2 \to X_2, B : X_2 \to X_1, C : X_1 \to X_2$. If T is a commutator, then $T = [T_1, T_2]$ for some $T_1, T_2 \in \mathcal{L}(\mathcal{X})$. Write

$T_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ for $i = 1, 2$. A simple computation shows that

$$T = \begin{pmatrix} [A_1, A_2] + B_1C_2 - B_2C_1 & A_1B_2 + B_1D_2 - A_2B_1 - B_2D_1 \\ C_1A_2 + D_1C_2 - C_2A_1 - D_2C_1 & [D_1, D_2] + C_1B_2 - C_2B_1 \end{pmatrix}.$$

From the fact that X_1 and X_2 have property **P**, and the fact that the B_1C_2, B_2C_1 lie in the intersection of all maximal ideals in $\mathcal{L}(X_1)$ and C_1B_2, C_2B_1 lie in the intersection of all maximal ideals in $\mathcal{L}(X_2)$ we immediately deduce that the diagonal entries in the last representation of T are commutators. In the preceding argument we used the fact that a perturbation of a commutator on a Banach space $\mathcal{Y}$ having property **P** by an operator that lies in the intersection of all maximal ideals in $\mathcal{L}(\mathcal{Y})$ is still a commutator. To show this fact assume that $A \in \mathcal{L}(\mathcal{Y})$ is a commutator, $B \in \mathcal{L}(\mathcal{Y})$ lies in the intersection of all maximal ideals in $\mathcal{L}(\mathcal{Y})$ and $A + B = \lambda I + S$ where S is an element of some ideal M in $\mathcal{L}(\mathcal{Y})$. Now using the simple observation that every ideal is contained in some maximal ideal, we conclude that $S - B$ is contained in a maximal ideal, say $\tilde{M}$ containing M hence $A - \lambda I \in \tilde{M}$, which is a contradiction with the assumption that $\mathcal{Y}$ has property **P**.

For the other direction we apply [**6**, Lemma 19] which concludes the proof in the case $n = 2$. The general case follows from the same considerations as in the case $n = 2$ in a obvious way. □

Our last corollary shows that property **P** is preserved under taking finite sums of Banach spaces having property **P** and some additional assumptions as in Lemma 5.3.

Corollary 5.4. *Let $\{X_i\}_{i=1}^n$ be a finite sequence of Banach spaces that have property* ***P****. Assume also that all operators $A\colon X_i \to X_i$ that factor through X_j are in the intersection of all maximal ideals in $\mathcal{L}(X_i)$ for each $i, j = 1, 2, \ldots, n,\ i \neq j$. Then $\mathcal{X} = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ has property* ***P****.*

Proof. Assume that $T \in \mathcal{L}(\mathcal{X})$ is not a commutator. Using Lemma 5.3, this can happen if and only if P_iTP_i is not commutator on X_i for some $i \in \{1, 2, \ldots, n\}$ and without loss of generality assume that $i = 1$. Since P_1TP_1 is not a commutator and X_1 has property **P** then $P_1TP_1 = \lambda I_{X_1} + S$ where S belongs to some maximal ideal J of $\mathcal{L}(X_1)$. Consider

$$M = \{B \in \mathcal{L}(\mathcal{X}) \,:\, P_1BP_1 \in J\}. \tag{5.2}$$

Clearly, if $B \in M$ and $A \in \mathcal{L}(\mathcal{X})$, then $AB, BA \in M$ because of the assumption on the operators from X_1 to X_1 that factor through X_j. It is also obvious that M is closed under addition, hence M is an ideal. Now it is easy to see that $T - \lambda I \in M$ which shows that all non-commutators have the form $\lambda I + S$, where $\lambda \neq 0$ and S belongs to some proper ideal of $\mathcal{L}(\mathcal{X})$.

The other direction follows from our comment in the beginning of the introduction that no operator of the form $\lambda I + S$ can be a commutator for any $\lambda \neq 0$ and any operator S which lies in a proper ideal of $\mathcal{L}(\mathcal{X})$.

□

References

1. F. Albiac and N. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics, 233. Springer, New York, 2006.
2. C. Apostol, 'On the left essential spectrum and non-cyclic operators in Banach spaces', *Rev. Roum. Math. Pures Appl.* 17 (1972) 1141–1148.
3. C. Apostol, 'Commutators on ℓ_p spaces', *Rev. Roum. Math. Pures* Appl. 17 (1972) 1513–1534.

4. C. Apostol, 'Commutators on c_0-spaces and on ℓ_∞-spaces', *Rev. Roum. Math. Pures Appl.* 18 (1973) 1025–1032.
5. A. Brown and C. Pearcy, 'Structure of commutators of operators', *Ann. of Math.* 82 (1965) 112–127.
6. D. Dosev 'Commutators on ℓ_1', *J. of Func. Analysis* 256 (2009) 3490–3509.
7. P. Enflo and T.W. Starbird, 'Subspaces of L_1 containing L_1', *Studia Math.* 65 (1979) 203–225.
8. I. A. Feldman, I. C. Gohberg and A. S. Markus, 'Normally solvable operators and ideals associated with them', *Izv. Moldavsk. Filial Akad. Nauk SSSR* 10 (76) (1960) 5169 (in Russian).
9. W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, 'Symmetric structures in Banach spaces', *Mem. Amer. Math. Soc.* 19 (1979), no. 217
10. J. Lindenstrauss, 'On complemented subspaces of m', *Israel J. Math.* 5 (1967) 153–156.
11. J. Lindenstrauss and A. Pełczyński, 'Contributions to the theory of the classical Banach spaces', *J. of Func. Analysis* 8 (1971) 225–249.
12. J. Lindenstrauss and H. P. Rosenthal, 'Automorphisms in c_0, ℓ_1 and m', *Israel J. Math.* 7 (1969) 227–239.
13. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 92 Springer-Verlag, Berlin-New York, 1977
14. A. Pełczyński, 'Projections in certain Banach spaces', *Studia Math.* 19 (1960) 209–228.
15. H. Porta, 'Factorable and strictly singular operators I', *Studia Math.* 37 (1971) 237–243.
16. H. P. Rosenthal, 'On relatively disjoint families of measures, with some applications to Banach space theory', *Studia Math.* 37 (1970) 13–36.
17. R. J. Whitley, 'Strictly singular operators and their conjugates', *Trans. Amer. Math. Soc.* 113 (1964) 252–261
18. A. Wintner, 'The unboundedness of quantum-mechanical matrices', *Phys. Rev.* 71 (1947) 738–739.

Detelin Dosev
Department of Mathematics
Texas A&M University
College Station, Texas 77843
USA

dossev@math.tamu.edu

William B. Johnson
Department of Mathematics
Texas A&M University
College Station, Texas 77843
USA

johnson@math.tamu.edu