COMPLEMENTABLY UNIVERSAL BANACH SPACES, II

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Abstract. The two main results are:

A. If a Banach space X is complementably universal for all subspaces of c_0 which have the bounded approximation property, then X^* is non separable (and hence X does not embed into c_0),

B. There is no separable Banach space X such that every compact operator (between Banach spaces) factors through X.

Theorem B solves a problem that dates from the 1970s.

1. Introduction

Given a class $\mathcal{O}$ of (bounded, linear) operators, it is natural to try to find a single (usually separable) Banach space U such that all the operators in $\mathcal{O}$ factor through U. In this case we say that $\mathcal{O}$ *factors through* U. We say that $\mathcal{O}$ λ*-factors through* U provided that for each $S \in \mathcal{O}$ there exist operators A,B such that $S = BA$, U is the co-domain of A and the domain of B, and $\|A\|\|B\| \leq \lambda$. If there is a λ so that the class $\mathcal{O}$ λ-factors through U, we say that $\mathcal{O}$ *uniformly factors through* U.

These concepts were, essentially, introduced by A. Pełczyński in [14]. He used the following definition: A Banach space U is said to be *complementably universal* for a class $\mathcal{B}$ of Banach spaces provided every space in $\mathcal{B}$ is isomorphic to a complemented subspace of U, i.e. if for every $B \in \mathcal{B}$, the identity on B factors through U. We shall also say that U is λ*-complementably universal for the class* $\mathcal{B}$ if for every $B \in \mathcal{B}$, the identity on B λ-factors through U.

For $\mathcal{B}_{bas}$ = the class of all separable Banach spaces that have a (Schauder) basis, there is such a separable U; namely, the separable

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universal basis space of [14]. Kadec [11] subsequently constructed a separable Banach space with the bounded approximation property (BAP) which is complementably universal for the class of all separable Banach spaces which have the BAP. Actually, the spaces constructed by Kadec and Pełczyński are isomorphic (see [8] and [15]).

In part one of this paper ([9]) we proved the non existence of a separable Banach space which is complementably universal for each of the following classes of Banach spaces:

1. $\mathcal{B}_{AP}$ = all separable Banach spaces which have the approximation property (AP),

2. $\mathcal{B}_p$ = all subspaces of ℓ_p for $2 < p < \infty$.

In particular, there is no separable Banach space which is complementably universal for the class of all separable Banach spaces.

In section 2 we first observe that for $1 < p < \infty$, there is a subspace of ℓ_p which is complementably universal for the class of all subspaces of ℓ_p which have the AP. We also note that there is a subspace of c_0 which is complementably universal for the class of all subspaces of c_0 whose duals have the AP. These observations, which are very simple given results from the 1970s, were known to the authors when [9] was written and likely are known to other old timers.

The first main result of this paper is Theorem 2.2, which says that if U is a Banach space which is complementably universal for the class of subspaces of c_0 which have BAP, then U^* is non separable (and hence U cannot be isomorphic to a subspace of c_0). This is done by using, as was done in [9], a variation of Davie's construction to produce a collection of subspaces of c_0 so that there is no separable Banach space which is complementably universal for their conjugate spaces. We then use an observation of Johnson and Schechtman, contained in [7], that a subspace X of c_0 is contained in another subspace Y of c_0 which has the BAP (even a finite dimensional decomposition) and such that X^* is isomorphic to a complemented subspace of Y^*.

Pełczyński's universal basis space U has the property that every operator that is uniformly approximable by finite rank operators factors through U. Many other spaces, including some separable reflexive spaces, [6], have the same property. All these results from the 1960s and 1970s left open the problem whether there is a separable Banach space Z so that every compact operator factors through Z. Our second main result, Theorem 2.5, is that there is no such space. We are indebted to Mariusz Wodzicki for reminding us in 1995 that whether such a space exists was still open.

We use standard Banach space theory terminology, as may be found in [12]. $\mathcal{K}$ denotes the class of all compact operators (between Banach spaces).

2. Results

We begin with a theorem which perhaps should be termed a "folklore result". It is at any rate a simple consequence of results proved in the 1970s and was known to the authors when [9] was written. First we set some notation. For $1 \le p \le \infty$, let $\{G_n^p\}_{n=1}^\infty$ be sequence of finite dimensional subspaces of ℓ_p such that for every $\epsilon > 0$, every finite dimensional subspace of ℓ_p is $1+\epsilon$–isomorphic to one (and hence infinitely many) of the spaces in $\{G_n^p\}_{n=1}^\infty$. Let Y_p be the ℓ_p sum of $\{G_n^p\}_{n=1}^\infty$ and Y_0 the c_0 sum of $\{G_n^0\}_{n=1}^\infty$.

Theorem 2.1. *(a) The space Y_p, $1 < p < \infty$, is complementably universal for the family of all subspaces of ℓ_p which have the approximation property.*
(b) The space Y_0 is complementably universal for the family of all subspaces of c_0 whose duals have the approximation property.

Proof: To prove (a), first fix $1 < p < \infty$ and let X be a subspace of ℓ_p which has a finite dimensional decomposition. Then by [10] (or see [12, Theorem 2.d.1]), X is isomorphic to the ℓ_p sum of a sequence of finite dimensional spaces. It follows from the construction of Y_p that X is isomorphic to a complemented subspace of Y_p. In the general case, where X is a subspace of ℓ_p which has the AP, by a theorem of Grothendieck [12, Theorem 1.e.15], X has the metric approximation property and hence the BAP . It then follows from the argument for Theorem 4 of [5] that $X \oplus_p Y_p$ is a Π–space; that is, that there exists a sequence of finite rank projections on $X \oplus_p Y_p$ which converges strongly to the identity operator. Since the dual of $X \oplus_p Y_p$ also has the BAP, Theorem 1.3 in [8] yields that $X \oplus_p Y_p$ has a finite dimensional decomposition. Therefore, by the first step of the proof, $X \oplus_p Y_p$ is isomorphic to a complemented subspace of Y_p.

The proof of (b) uses the same ingredients. If X is a subspace of c_0 which has a shrinking finite dimensional decomposition, then again by [9] (or see [12, Theorem 2.d.1]), X is isomorphic to the c_0 sum of a sequence of finite dimensional spaces and hence is isomorphic to a complemented subspace of Y_0. If X is a subspace of c_0 whose dual has the approximation property, then Grothendieck's theorem [12, Theorem 1.e.15] implies that X^* has the BAP. One then uses [5] and [8] in the same way as in the ℓ_p case to conclude that X is isomorphic to a complemented subspace of Y_0. ■

Theorem 2.1 suggests the following problem: What can one say about a separable Banach space which is complementably universal for the collection of all subspaces of c_0 which have the BAP? By the results of Kadec [11] and Pełczyński [14] mentioned in the introduction, such spaces do exist. The most natural question is whether a subspace of c_0 can have this universal property. One of the main results of this note gives a negative answer to this question:

Theorem 2.2. *Let U be a Banach space which is complementably universal for the family of all subspaces of c_0 which have the BAP. Then U^* is non separable.*

The main technical tool for proving Theorem 2.2 is Theorem 2.3.

Theorem 2.3. *There is no separable Banach space which is complementably universal for the family $\mathcal{D}_0$ of duals to subspaces of c_0.*

Let us observe here that, since $\mathcal{D}_0$ is closed under l_1-sums, by Proposition 2 in [6] it suffices to prove the following statement:

Theorem 2.4. *There is no separable Banach space which is uniformly complementably universal for $\mathcal{D}_0$.*

Theorem 2.4 is proved in section 4 . The proof is similar to the proof in [9] that no separable Banach space is complementably universal for the collection of subspaces of ℓ_p when $2 < p < \infty$, and indeed the argument in section 4 gives this result from [9]. Since the argument we need here is more involved, we chose to give a detailed, complete and streamlined proof of Theorem 2.4.

Once Theorem 2.3 is known, we complete the proof of Theorem 2.2 with the following proposition which is proved but not stated in [7, section 2].

Proposition 1. *Let X be a subspace of c_0. Then there is a subspace Y of c_0 which has a finite dimensional decomposition and such that X^* is isomorphic to a complemented subspace of Y^*.*

It is obvious that Theorem 2.2 follows from the conjunction of Theorem 2.3 and Proposition 1. Here we repeat part of the discussion in [7, section 2] which yields Proposition 1 and refer to [7] for additional details. Let $E_1 \subset E_2 \subset \dots$ be a sequence of finite dimensional subspaces of X whose union is dense in X and let Y be the subspace of the ℓ_∞ sum $(\sum_n E_n)_\infty$ of $\{E_n\}_{n=1}^\infty$ consisting of sequences $(e_1, e_2, \dots)$ for which $\lim_{n\to\infty} e_n$ exists in X. The space Y has a monotone finite dimensional decomposition.. Indeed, for each positive integer n define a contractive projection P_n on Y by setting

$P(e_1, e_2, \dots) = (e_1, e_2, \dots, e_{n-1}, e_n, e_n, \dots)$. It is easy to check that $\{P_n\}_{n=1}^{\infty}$ is the sequence of partial sum projections for a finite dimensional decomposition of Y. Now define an operator Q from Y into X by setting $Q(e_1, e_2, \dots) = \lim_n e_n$. It is easy to check that Q is a quotient mapping from Y onto X with kernel the c_0 sum of $\{E_n\}_{n=1}^{\infty}$. This is a construction used to good effect by Lusky [13]. The main new points in [7] are the observations that the separable injectivity of c_0 yields that Y is isomorphic to a subspace of c_0, and that, by [6], the range of Q^* (which is isometric to X^*) is norm one complemented in Y^*. This completes the proof of Proposition 1 and hence also the proof of Theorem 2.2.

We now state our second main result.

Theorem 2.5. *The class of all compact operators (between Banach spaces) does not factor through a separable Banach space.*

Evidently, it is enough to prove the following

Theorem 2.6. *The class of all compact operators of norm ≤ 1 does not uniformly factor through a separable Banach space.*

The proofs of Theorems 2.4 and 2.6 are based on a construction which has two essential components: linear-algebraic, described in section 5 and probabilistic, described in section 6.

3. Preliminaries

We shall use the following lemma, which is a refinement of Lemma 2 in [9]:

Lemma 1. *Let $\mathcal{A}$ be an uncountable set, let B, Z be Banach spaces and let $D_{\boldsymbol{\varepsilon}}$ be a subspace of B, $R_{\boldsymbol{\varepsilon}}$ be a subspace of Z for $\boldsymbol{\varepsilon} \in \mathcal{A}$. Suppose that there are $\delta > 0, \lambda > 0$ and a finite dimensional space E with $E \subset \bigcap_{\boldsymbol{\varepsilon} \in \mathcal{A}} D_{\boldsymbol{\varepsilon}}, E \subset \bigcap_{\boldsymbol{\varepsilon} \in \mathcal{A}} R_{\boldsymbol{\varepsilon}}$, so that the following condition is satisfied:*

($\sharp_1$) *if $\boldsymbol{\varepsilon}, \boldsymbol{\eta} \in \mathcal{A}$, $\boldsymbol{\varepsilon} \neq \boldsymbol{\eta}$ and $T : D_{\boldsymbol{\varepsilon}} \to R_{\boldsymbol{\eta}}$ is such that $\|T_{|E} - Id_E\| < \delta$, then $\|T\| > \lambda$.*

Let $T_{\boldsymbol{\varepsilon}} : D_{\boldsymbol{\varepsilon}} \to R_{\boldsymbol{\varepsilon}}$ be bounded operators such that $T_{\boldsymbol{\varepsilon}|E} = Id_E$ for every $\boldsymbol{\varepsilon} \in \mathcal{A}$. Then the family of operators $\{T_{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \in \mathcal{A}\}$ does not λ-factor through a separable Banach space.

Proof. Suppose U is a separable Banach space such that for every $\boldsymbol{\varepsilon} \in \mathcal{A}$, the operator $T_{\boldsymbol{\varepsilon}}$ has a factorization $T_{\boldsymbol{\varepsilon}} = Q_{\boldsymbol{\varepsilon}} S_{\boldsymbol{\varepsilon}}$ with $S_{\boldsymbol{\varepsilon}} : D_{\boldsymbol{\varepsilon}} \to U$ and $Q_{\boldsymbol{\varepsilon}} : U \to R_{\boldsymbol{\varepsilon}}$ so that $\|S_{\boldsymbol{\varepsilon}}\| \leq \lambda, \|Q_{\boldsymbol{\varepsilon}}\| \leq 1$. Since the space $L(E, U)$ is separable and $\mathcal{A}$ is uncountable, there are $\boldsymbol{\varepsilon}, \boldsymbol{\eta} \in \mathcal{A}, \boldsymbol{\varepsilon} \neq \boldsymbol{\eta}$

such that $\|(S_{\boldsymbol{\varepsilon}} - S_{\boldsymbol{\eta}})_{|E}\| < \delta$. Let us define $T = Q_{\boldsymbol{\varepsilon}} S_{\boldsymbol{\eta}}$. We have $(Q_{\boldsymbol{\varepsilon}} S_{\boldsymbol{\varepsilon}})_{|E} = Id_E$, thus

$$T_{|E} - Id_E = Q_{\boldsymbol{\varepsilon}}(S_{\boldsymbol{\eta}} - S_{\boldsymbol{\varepsilon}})_{|E}$$

and therefore $\|T_{|E} - Id_E\| \le \|Q_{\boldsymbol{\varepsilon}}\|\|(S_{\boldsymbol{\varepsilon}} - S_{\boldsymbol{\eta}})_{|E}\| < \delta$. Therefore, by $(\sharp_1)$, $\|T\| > \lambda$. This is a contradiction with

$$\|T\| \le \|Q_{\boldsymbol{\varepsilon}}\|\|S_{\boldsymbol{\eta}}\| \le \lambda.$$

■

The next lemma is a "complementably universal" version of Lemma 1:

Lemma 2. *Let $\mathcal{A}$ be an uncountable set, let B be a Banach space and let $B_{\boldsymbol{\varepsilon}}$ be a subspace of B for $\boldsymbol{\varepsilon} \in \mathcal{A}$. Suppose that there are $\delta > 0, \lambda > 0$ and a finite dimensional space E, $E \subset \bigcap_{\boldsymbol{\varepsilon} \in \mathcal{A}} B_{\boldsymbol{\varepsilon}}$ so that the following condition is satisfied:*

$(\sharp_2)$ *if $\boldsymbol{\varepsilon}, \boldsymbol{\eta} \in \mathcal{A}$, $\boldsymbol{\varepsilon} \ne \boldsymbol{\eta}$ and $T : B_{\boldsymbol{\varepsilon}} \to B_{\boldsymbol{\eta}}$ is such that $\|T_{|E} - Id_E\| < \delta$, then $\|T\| > \lambda$.*

Then there is no separable Banach space which is λ-complementably universal for the family $\{B_{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \in \mathcal{A}\}$.

The construction of appropriate families $\{B_{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \in \mathcal{A}\}$ (for Theorems 2.2 and 2.3) and $\{D_{\boldsymbol{\varepsilon}}, R_{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \in \mathcal{A}\}$ (for Theorems 2.5 and 2.6) has two essential components: a linear-algebraic one (section 5) and a probabilistic one (section 6).

Now we would like to make explicit some tensor product notation.

Given finite sets $I \subset J$, let Π_I denote the coordinate projection in $\mathbb{R}^J$ onto the coordinates in I.

Let E, F be vector spaces, let $e_1, \dots, e_M; f_1, \dots, f_N$ be bases in E, F, respectively. Let $I \subset J$ be finite sets and let $x_i \in E,\ w_i \in F$ for $i \in J$ with

$$x_i = \sum_{m=1}^{M} x(i,m) e_m, w_i = \sum_{n=1}^{N} w(i,n) f_n.$$

Then

$$\sum_{i \in I} w_i \otimes x_i = \sum_{n=1}^{N} f_n \otimes X^* \Pi_I W(f_n) \tag{1}$$

where, formally, $X : E \to \mathbb{R}^J$ and $W : F \to \mathbb{R}^J$ are defined by $Xe_m = (x(i,m))_{i \in J}$ for $m = 1, \dots, M$ and $Wf_n = (w(i,n))_{i \in J}$ for $n = 1, \dots, N$.

Let Z, Y be Banach spaces. For $\beta = \Sigma \psi_i \otimes z_i \in Y^* \otimes Z$ let us denote supp β = span $\{z_i\}$ and $\beta^\perp = \bigcap \psi_i^\perp = \{y \in Y : \psi_i(y) = 0 \text{ for all } i\}$

(these notions depend on a specific representation of the tensor β but this will not interfere with our considerations).

As usual, a $\beta \in Y^* \otimes Z$ acts on $L(Z', Y')$ for any $Z' \supset \operatorname{supp} \beta$ and $Y' \subset Y$ by the formula $\beta(T) = \Sigma\psi_i(Tz_i)$. For such a T we have

$$|\beta(T)| \leq \|\beta\|_{Y^*\hat{\otimes}Z'} \|T\|_{L(Z',Y')}.$$

(Recall that

$$\|\beta\|_{Y^*\hat{\otimes}Z'} = \inf\{\Sigma\|\psi_i\|\|z_i\| : \beta = \Sigma\psi_i \otimes z_i \text{ with } \psi_i \in Y^*, z_i \in Z'\}.)$$

Lemma 3. *Let Z, Y be Banach spaces, let $E \subset Z \cap Y$. Let $\beta, \gamma \in Y^* \otimes Z$. Assume that $E \supset \operatorname{supp} \beta$ and $\beta(Id_E) = 1$. Suppose that $T : Z' \to Y'$ where $Y' \subset \gamma^\perp$, $Z' \supset \operatorname{supp} \beta + \operatorname{supp} \gamma$. Then*

$$\|T\| \geq \|\beta - \gamma\|^{-1}_{Y^*\hat{\otimes}Z'}(1 - \|\beta\|_{Y^*\hat{\otimes}Z'}\|T_{|E} - Id_E\|).$$

Proof. Since the range of T is $\gamma^\perp$, $\gamma(T) = 0$ and thus

$$(\beta - \gamma)(T) = \beta(T) = 1 + \beta(T_{|E} - Id_E).$$

Hence

$$\|\beta - \gamma\|_{Y^*\hat{\otimes}Z'}\|T\| \geq 1 - \|\beta\|_{Y^*\hat{\otimes}Z'}\|T_{|E} - Id_E\|,$$

which proves the lemma. ∎

4. The proofs

In this section we give a schematic outline of the proofs of Theorems 2.4 and 2.6 based on Lemmas 2 and 1. Technical details of the construction are deferred to sections 5 and 6.

Let $\mathbb{R}^\infty = \{(x_k)_{k=0}^\infty : x_k \in \mathbb{R}^{3\cdot 2^k}, x_k \neq 0 \text{ for finitely many } k\}$. In $\mathbb{R}^\infty$ we have the natural inner product $\langle (x_k), (z_k)\rangle = \sum_{k=0}^\infty \langle x_k, z_k\rangle$.

Let $X = (\Sigma\ell_1^{3\cdot 2^k})_{c_0}, Z = (\Sigma\ell_\infty^{3\cdot 2^k})_{\ell_1}$.

Let e_i^k for $k = 1, 2, \ldots; i = 1, \ldots, 3 2^k$ denote the unit vectors in $\mathbb{R}^\infty$.

In section 5 we shall describe vectors $y_i^k \in \mathbb{R}^\infty$ for $k = 1, 2, \ldots,$ $i = 1, 2, \ldots, 2^k$. These vectors satisfy the following conditions:

$$\langle y_i^k, y_j^m\rangle = 2\delta_{km}\delta_{ij} \quad \text{for all } i, j, k, m, \tag{2}$$

$$\|y_i^k\|_Z \leq 2 \cdot 2^{-\frac{k}{2}}, \quad \|y_i^k\|_X \leq 3 \cdot 2^{\frac{k}{2}}. \tag{3}$$

Let $Y = \operatorname{span}\{y_i^k : k = 1, 2, \ldots; i = 1, \ldots, 2^k\}$. Given an $x \in \mathbb{R}^\infty$, we identify it with the functional on Y defined by $y \rightsquigarrow \langle y, x\rangle$.

Let us denote for $k = 1, 2, \ldots$:

$$I_0^k = \{1, \ldots, 2^{k-1}\}, I_1^k = \{2^{k-1} + 1, \ldots, 2^k\}.$$

Let us define for $k = 1, 2, \ldots$ and $\varepsilon = 0, 1$:

$$Y_\varepsilon^k = \operatorname{span}\{y_i^k : i \in I_\varepsilon^k\}$$

and

$$\beta_\varepsilon^k = 2^{-k} \sum_{i \in I_\varepsilon^k} y_i^k \otimes y_i^k \in Y^* \otimes Y_\varepsilon^k \subset Y^* \otimes Y. \tag{4}$$

Observe that, by (2),

$$\beta_\varepsilon^k(Id_{Y_\varepsilon^k}) = 1 \text{ for every } k, \varepsilon. \tag{5}$$

Let us denote $\mathcal{C} = \{0,1\}^{\mathbb{N}}$. For $\boldsymbol{\varepsilon} \in \mathcal{C}$ let $Y_{\boldsymbol{\varepsilon}} = \operatorname{span} \bigcup_{k=1}^\infty Y_{\boldsymbol{\varepsilon}(k)}^k$. The family of spaces $\{Y_{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \in \mathcal{C}\}$ will be the basis for our constructions.

In section 5 we shall obtain the following key representation for the difference $\beta_\varepsilon^k - \beta_\eta^{k+1} \in Y^* \otimes Y$:

$$\beta_\varepsilon^k - \beta_\eta^{k+1} = 2^{-k} \sum_{j=1}^{2^k} e_j^k \otimes d_{\varepsilon,\eta,j}^k \tag{6}$$

where $d_{\varepsilon,\eta,j}^k$ for $k = 1, 2, \ldots ; j = 1, \ldots 2^k$ and $\varepsilon, \eta = 0, 1$ satisfy the following two conditions:

$$\|d_{\varepsilon,\eta,j}^k\|_Z \leq Ck^2 2^{-\frac{k}{2}}, \tag{7}$$

the constant C being independent of k, j, ε, η, and

$$d_{\varepsilon,\eta,j}^k \in Y_\varepsilon^k + Y_\eta^{k+1}. \tag{8}$$

Proof of Theorem 2.4. With the duality generated by the inner product $\langle , \rangle$, we have $Z = X^*$ (and, obviously, $X \subset Z^*$).

For $A \subset Z$ denote $A_\perp = \{x \in X : \langle z, x \rangle = 0 \text{ for all } z \in A\}$ and for $A \subset X$ denote $A^\perp = \{z \in Z : \langle z, x \rangle = 0 \text{ for all } x \in A\}$.

Let $B = (Y_\perp)^\perp$ and let $B_{\boldsymbol{\varepsilon}} = ((Y_{\boldsymbol{\varepsilon}})_\perp)^\perp$ for $\boldsymbol{\varepsilon} \in \mathcal{C}$. Evidently B and $B_{\boldsymbol{\varepsilon}}$ are w*-closed subspaces of $Z = X^*$, $B_{\boldsymbol{\varepsilon}} \subset B$ for every $\boldsymbol{\varepsilon} \in \mathcal{C}$ and we have

$$(Y_{1-\boldsymbol{\varepsilon}(k)}^k)^\perp \supset B_{\boldsymbol{\varepsilon}} \supset Y_{\boldsymbol{\varepsilon}(k)}^k \text{ for every } k.$$

Define β_ε^k by (4). We have $\|y_i^k\|_{Z^*} = \|y_i^k\|_X$, thus, by (3),

$$\|y_i^k\|_{Z^*} \|y_i^k\|_Z \leq 6 \text{ for all } i, k.$$

Hence (here Y_ε^k is equipped with the norm of Z):

$$\|\beta_\varepsilon^k\|_{B^* \hat{\otimes} Y_\varepsilon^k} \leq 6 \text{ for all } \varepsilon, k. \tag{9}$$

Since $\|e_j^k\|_{Z^*} = 1$, by the representation (6), we obtain for every k and $\boldsymbol{\varepsilon} \in \mathcal{C}$,

$$\|\beta_{\boldsymbol{\varepsilon}(k)}^k - \beta_{\boldsymbol{\varepsilon}(k+1)}^{k+1}\|_{B^* \hat{\otimes} B_{\boldsymbol{\varepsilon}}} \leq Ck^2 2^{-\frac{k}{2}}, \tag{10}$$

the constant C being independent of k and $\boldsymbol{\varepsilon} \in \mathcal{C}$. Therefore for every $k \leq m$,

$$\|\beta^k_{\boldsymbol{\varepsilon}(k)} - \beta^m_{\boldsymbol{\varepsilon}(m)}\|_{B^* \hat{\otimes} B_{\boldsymbol{\varepsilon}}} \leq C k^2 2^{-\frac{k}{2}} \tag{11}$$

with another constant C.

Applying Lemma 2 and Lemma 3 we can now prove Theorem 2.4. Indeed, denote $\mathcal{C}_k = \{\boldsymbol{\varepsilon} \in \mathcal{C} : \boldsymbol{\varepsilon}(1) = \cdots = \boldsymbol{\varepsilon}(k) = 0\}$, let $E = E_k = Y_0^k = \operatorname{span}\{y_i^k : 1 \leq i \leq 2^{k-1}\}$.

Let $\boldsymbol{\varepsilon}, \boldsymbol{\eta} \in \mathcal{C}_k, \boldsymbol{\varepsilon} \neq \boldsymbol{\eta}$, let m be the first index larger than k such that $\boldsymbol{\varepsilon}(m) \neq \boldsymbol{\eta}(m)$, thus $R_{\boldsymbol{\eta}} \subset \beta^{\perp}_{\boldsymbol{\varepsilon}(m)}$. Let $T : B_{\boldsymbol{\varepsilon}} \to B_{\boldsymbol{\eta}}$ be such that $\|T_{|E} - Id_E\| < \delta = \frac{1}{12}$.

Observe that, by (5), $\beta^k_{\boldsymbol{\varepsilon}(k)}(Id_E) = 1$ for every $\boldsymbol{\varepsilon} \in \mathcal{C}_k$. By Lemma 3, (9) and (11),

$$\|T\| \geq \|\beta^k_{\boldsymbol{\varepsilon}(k)} - \beta^m_{\boldsymbol{\varepsilon}(m)}\|^{-1}_{B^* \hat{\otimes} B_{\boldsymbol{\varepsilon}}} (1 - \|\beta^k_{\boldsymbol{\varepsilon}(k)}\|_{B^* \hat{\otimes} B_{\boldsymbol{\varepsilon}}} \delta) > \lambda_k = \frac{1}{2C} k^{-2} 2^{\frac{k}{2}}.$$

Since $T_{\boldsymbol{\varepsilon}|E} = I_E$, by Lemma 2, there is no separable Banach space which is λ_k-complementably universal for the family $\{B_{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \in \mathcal{C}_k\}$. (Perhaps it looks a bit strange that we can prove that $\lambda_1, \lambda_2, \lambda_3, \ldots$ increase while $\mathcal{C}_1 \supset \mathcal{C}_2 \supset \ldots$. The reason is that, by the estimate of Lemma 2, λ_k increase together with the size of E_k and the spaces E_k do grow.) Consequently, there is no separable Banach space which is uniformly complementably universal for the family $\{B_{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \in \mathcal{C}\}$.

To conclude, let us observe that each $B_{\boldsymbol{\varepsilon}}$ is isomorphic to the dual of a subspace of c_0: being w*-closed in Z, $B_{\boldsymbol{\varepsilon}}$ is isometric to the dual of a quotient of X and X is clearly isomorphic to a subspace of c_0. Finally, every subspace of a quotient of c_0 is isomorphic to a subspace of c_0 (see [10] or [1]).

Proof of Theorem 2.6. In Y we will define a norm $|||\ |||$ in the following way. Let

$$V = \{y_i^k : k = 1, 2, \ldots; i = 1, \ldots, 2^k\},$$
$$W = \{2^{\frac{k}{4}} d^k_{\varepsilon,\eta,i} : k = 1, 2, \ldots, \varepsilon, \eta = 0, 1, i = 1, \ldots, 2^k\},$$

let $U = \operatorname{absconv}(V \cup W)$ and let for $x \in Y$

$$|||x||| = \inf\{\lambda : x \in \lambda U\}.$$

Let B be the completion of Y in the norm $|||\ |||$, let $T : B \to Z$ be the completion of Id_Y.

It is clear that T has norm ≤ 1. We also see that T is a compact operator, since U is dense in the unit ball of B and $T(U)$ is the convex hull of a sequence which converges to 0 in Z (observe that, by (7), $\|2^{\frac{k}{4}} d^k_{\varepsilon,\eta,i}\|_Z \le Ck^2 2^{-\frac{k}{4}}$).

Let $D_{\boldsymbol{\varepsilon}}$ be the closure of $Y_{\boldsymbol{\varepsilon}}$ in B, let $R_{\boldsymbol{\varepsilon}}$ be the closure of $Y_{\boldsymbol{\varepsilon}}$ in Z and let $T_{\boldsymbol{\varepsilon}} = T_{|D_{\boldsymbol{\varepsilon}}}$.

Now the argument goes like that of the previous proof:

Define β^k_ε by (4). We have $\|y^k_i\|_{B^*} = \|y^k_i\|_X \le 2 \cdot 2^{\frac{k}{2}}$ and $\|y^k_i\|_{D_{\boldsymbol{\varepsilon}}} \le 1$, therefore

$$\|\beta^k_{\boldsymbol{\varepsilon}(k)}\|_{B^* \hat{\otimes} D_{\boldsymbol{\varepsilon}}} \le 2 \cdot 2^{\frac{k}{2}} \text{ for all } \boldsymbol{\varepsilon}, k. \tag{12}$$

Let us use the representation $\beta^k_\varepsilon - \beta^{k+1}_\eta = 2^{-k} \sum_{j=1}^{2^k} e^k_j \otimes d^k_{\varepsilon,\eta,j}$ (cf.(6)). We have $\|e^k_j\|_{Z^*} = 1$ and, by the definition of the norm $|||\ |||$, we have $|||d^k_{\boldsymbol{\varepsilon}(k),\boldsymbol{\varepsilon}(k+1),i}||| \le 2^{-\frac{k}{4}}$, therefore

$$\|\beta^k_{\boldsymbol{\varepsilon}(k)} - \beta^{k+1}_{\boldsymbol{\varepsilon}(k+1)}\|_{B^* \hat{\otimes} D_{\boldsymbol{\varepsilon}}} \le 2^{-\frac{k}{4}}.$$

Consequently, for every $m > k$

$$\|\beta^k_{\boldsymbol{\varepsilon}(k)} - \beta^m_{\boldsymbol{\varepsilon}(m)}\|_{B^* \hat{\otimes} D_{\boldsymbol{\varepsilon}}} \le 6 \cdot 2^{-\frac{k}{4}}. \tag{13}$$

Let again $\mathcal{C}_k = \{\boldsymbol{\varepsilon} \in \mathcal{C} : \boldsymbol{\varepsilon}(1) = \cdots = \boldsymbol{\varepsilon}(k) = 0\}$ and $E = Y^k_0$. Let us take $\delta = \frac{1}{4} 2^{-\frac{k}{2}}$. Let $\boldsymbol{\varepsilon}, \boldsymbol{\eta} \in \mathcal{C}_k, \boldsymbol{\varepsilon} \ne \boldsymbol{\eta}$, let m be the first index larger than k such that $\boldsymbol{\varepsilon}(m) \ne \boldsymbol{\eta}(m)$, thus $R_{\boldsymbol{\eta}} \subset \beta^{\perp}_{\boldsymbol{\varepsilon}(m)}$. Let $T : D_{\boldsymbol{\varepsilon}} \to R_{\boldsymbol{\eta}}$ be such that $\|T_{|E} - Id_E\| < \delta$. By Lemma 3, (12) and (13)

$$\|T\| \ge \|\beta^k_{\boldsymbol{\varepsilon}(k)} - \beta^m_{\boldsymbol{\varepsilon}(m)}\|^{-1}_{B^* \hat{\otimes} D_{\boldsymbol{\varepsilon}}} (1 - \|\beta^k_{\boldsymbol{\varepsilon}(k)}\|_{B^* \hat{\otimes} D_{\boldsymbol{\varepsilon}}} \delta) > 2^{\frac{k}{4} - 4}.$$

Since $T_{\boldsymbol{\varepsilon}|E} = Id_E$, by Lemma 1, the family $\{T_{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \in \mathcal{C}_k\}$ does not $2^{\frac{k}{4}-4}-$ uniformly factor through a separable Banach space, hence the family $\{T_{\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} \in \mathcal{C}\}$ does not uniformly factor through a separable Banach space.

5. The Construction of y^k_i

Let us denote

$$I^k_0 = \{1, \ldots, 2^{k-1}\}, I^k_1 = \{2^{k-1} + 1, \ldots, 2^k\}.$$

For $k = 0, 1, 2, \ldots,$ let $u^k_1, \ldots, u^k_{3 \cdot 2^k}$, be orthonormal systems in $\mathbb{R}^{3 \cdot 2^k}$. For $j = 1, \ldots, 2^{k+1}$ denote $v^k_j = u^k_{2^k + j}$ and let us define $y^k_i \in \mathbb{R}^\infty$ for $k = 1, 2, \ldots; i = 1, 2, \ldots, 2^k$ by

$$y^k_i = v^{k-1}_i + u^k_i.$$

Due to the orthonormality of the whole system $\{u_i^k : k = 0,1,2,\ldots; i = 1,\ldots,3{\cdot}2^k\}$, we have

$$\langle y_i^k, y_j^m\rangle = 2\langle u_i^k, y_j^m\rangle = 2\langle v_i^{k-1}, y_j^m\rangle = 2\delta_{km}\delta_{ij} \quad \text{for all } i,j,k,m,$$

therefore we have for every $y \in Y (= \operatorname{span} y_i^k)$

$$\langle y_i^k, y\rangle = 2\langle u_i^k, y\rangle = 2\langle v_i^{k-1}, y\rangle, \tag{14}$$

i.e. for every $x \in Y$ the following tensors are identical in $Y^* \otimes Y$:

$$y_i^k \otimes x = 2u_i^k \otimes x = 2v_i^{k-1} \otimes x.$$

This is a crucial observation which allows us to represent $\beta_\varepsilon^k \in Y^* \otimes Y$ in two different ways:

$$\beta_\varepsilon^k = 2^{-k} \sum_{i \in I_\varepsilon^k} u_i^k \otimes y_i^k$$

and

$$\beta_\varepsilon^k = 2^{-k} \sum_{i \in I_\varepsilon^k} v_i^{k-1} \otimes y_i^k.$$

Now we shall transform these formulas, writing β_ε^k in terms of e_j^k, like in (1). Let

$$u_j^k = \sum_{j=1}^{3\cdot 2^k} u^k(i,j) e_j^k \text{ for } j = 1,\ldots,2^k, v_j^k = \sum_{j=1}^{3\cdot 2^k} v^k(i,j) e_j^k \text{ for } j = 1,\ldots,2^{k+1}.$$

We have for $I \subset \{1,\ldots,2^k\}$:

$$\sum_{i\in I} u_i^k \otimes y_i^k = \sum_{i\in I} u_i^k \otimes v_i^{k-1} + \sum_{i \in I} u_i^k \otimes u_i^k =$$
$$\sum_{j=1}^{3\cdot 2^k} e_j^k \otimes (V^{k-1})^* \Pi_I U^k(e_j^k) + \sum_{j=1}^{3\cdot 2^k} e_j^k \otimes (U^k)^* \Pi_I U^k(e_j^k)$$

and

$$\sum_{i\in I} v_i^{k-1} \otimes y_i^k = \sum_{i\in I} v_i^{k-1} \otimes v_i^{k-1} + \sum_{i\in I} v_i^{k-1} \otimes u_i^k =$$
$$\sum_{j=1}^{3\cdot 2^{k-1}} e_j^{k-1} \otimes (V^{k-1})^* \Pi_I V^{k-1}(e_j^{k-1}) + \sum_{j=1}^{3\cdot 2^{k-1}} e_j^k \otimes (U^k)^* \Pi_I V^{k-1}(e_j^{k-1}).$$

where $U^k : \mathbb{R}^{3\cdot 2^k} \to \mathbb{R}^{2^k}$ and $V^k : \mathbb{R}^{3\cdot 2^k} \to \mathbb{R}^{2^{k+1}}$ are defined by:

$$U^k e_j^k = (u^k(i,j))_{i=1}^{2^k} \text{ and } V^k e_j^k = (v^k(i,j))_{i=1}^{2^{k+1}}. \tag{15}$$

Denote now

$$U_\varepsilon^k = \Pi_{I_\varepsilon^k} U^k, \quad V_\varepsilon^k = \Pi_{I_\varepsilon^{k+1}} V^k. \tag{16}$$

We have for $\varepsilon, \eta = 0, 1$:

$$\begin{aligned}\beta_\varepsilon^k - \beta_\eta^{k+1} &= 2^{-k} \sum_{i \in I_\varepsilon^k} u_i^k \otimes y_i^k - 2^{-k-1} \sum_{i \in I_\eta^{k+1}} v_i^k \otimes y_i^{k+1} \\ &= 2^{-k-1} \sum_{j=1}^{2^k} e_j^k \otimes D_{\varepsilon,\eta}^k e_j^k\end{aligned}$$

where the map $D_{\varepsilon,\eta}^k : \mathbb{R}^{2^k} \to Y$ is defined by

$$D_{\varepsilon,\eta}^k = 2(V_\varepsilon^{k-1})^* U_\varepsilon^k + 2(U_\varepsilon^k)^* U_\varepsilon^k - (V_\eta^k)^* V_\eta^k - (U_\eta^{k+1})^* V_\eta^k.$$

Denote

$c_1 = (V_\varepsilon^{k-1})^* U_\varepsilon^k (e_j^k), c_2 = [2(U_\varepsilon^k)^* U_\varepsilon^k - (V_\eta^k)^* V_\eta^k](e_j^k), c_3 = (U_\eta^{k+1})^* V_\eta^k (e_j^k).$

Let $d_{\varepsilon,\eta,j}^k = D_{\varepsilon,\eta}^k e_j^k = c_1 + c_2 + c_3$. It is clear that $c_1 \in \mathbb{R}^{2\cdot 3^{k-1}}, c_2 \in \mathbb{R}^{2\cdot 3^k}, c_3 \in \mathbb{R}^{2\cdot 3^{k+1}}$, therefore

$$\|d_{\varepsilon,\eta,j}^k\|_Z = \|c_1\|_\infty + \|c_2\|_\infty + \|c_3\|_\infty.$$

For a matrix $Q = \{q(i,j)\}$ we denote $\|Q\|_\infty = \max |q(i,j)|$. Since c_1, c_2, c_3 are columns of the corresponding matrices, we have :

$$\begin{aligned}&\|c_1\|_\infty \le 2\|(V_\varepsilon^{k-1})^* U_\varepsilon^k\|_\infty, \|c_2\|_\infty \le \|2(U_\varepsilon^k)^* U_\varepsilon^k - (V_\eta^k)^* V_\eta^k\|_\infty, \\ &\|c_3\|_\infty \le \|(U_\eta^{k+1})^* V_\eta^k\|_\infty.\end{aligned}$$

The last section of this paper is devoted to proving that there exist orthonormal systems $u_1^k, \dots, u_{3\cdot 2^k}^k$ in $\mathbb{R}^{3\cdot 2^k}$ so that for all k, j, ε, η we have

$$\begin{aligned}&\|(V_\varepsilon^{k-1})^* U_\varepsilon^k\|_\infty \le Ck^2 2^{-\frac{k}{2}}, \|2(U_\varepsilon^k)^* U_\varepsilon^k - (V_\eta^k)^* V_\eta^k\|_\infty \le Ck^2 2^{-\frac{k}{2}}, \\ &\|(U_\eta^{k+1})^* V_\eta^k\|_\infty \le Ck^2 2^{-\frac{k}{2}},\end{aligned} \tag{17}$$

which clearly implies our key estimate (7). The proof is probabilistic: it turns out that choosing the system $u_1^k, \dots, u_{3\cdot 2^k}^k$ in $\mathbb{R}^{3\cdot 2^k}$ randomly (w.r.t. the Haar measure on the orthogonal group), c_1, c_2, c_3 will be small (i.e. of order $k^2 2^{-\frac{k}{2}}$) with large probability and therefore we can find such systems so that (17) is satisfied.

Let us observe here that the reasons for the smallness of c_1, c_2, c_3 are somewhat different: c_1 and c_3 are small because the matrices U_ε^k and V_η^{k-1} are independent and therefore all the entries of $(V_\eta^{k-1})^* U_\varepsilon^k$ are small. On the other hand, in the matrices $(U_\varepsilon^k)^* U_\varepsilon^k$ and $(V_\eta^k)^* V_\eta^k$ the diagonal elements are about 1/4, respectively 1/2 and the off-diagonal elements are small (because of independence). Consequently, all the elements of $2(U_\varepsilon^k)^* U_\varepsilon^k - (V_\eta^k)^* V_\eta^k$ are small (the diagonal elements cancel out) and this is why c_2 is small.

6. A probabilistic lemma

For a matrix $Q = \{q(i,j)\}$ we denote $\|Q\|_\infty = \max |q(i,j)|$.

Lemma 4. *For $j = 1, 2$, let Q_j be an $n \times m_j$ matrix, with $m_1, m_2 \le 4n$. Let n_1, n_2, n_3, n_4 be natural numbers so that $n = n_1 + n_2 + n_3 + n_4$. Then there exist disjoint sets $I_1, I_2, I_3, I_4 \subset \{1, \ldots, n\}$ with $\#I_\alpha = n_\alpha, \alpha = 1, 2, 3, 4$, such that for α, β in $\{1, 2, 3, 4\}$,*

$$\|Q_1^*(p_\beta \Pi_{I_\alpha} Q_2 - p_\alpha \Pi_{I_\beta} Q_2)\|_\infty \le C \|Q_1\|_\infty \|Q_2\|_\infty n^{1/2} (\log n)^{1/2}, \tag{18}$$

where $p_\alpha = \frac{n_\alpha}{n}$ for $\alpha = 1, 2, 3, 4$.

Remark. Instead of 4 one can take here any fixed natural number.

Proof. Let $X_1, \ldots, X_n$ be i.i.d. variables taking values 1,2,3,4 with probabilities p_1, p_2, p_3, p_4, respectively. The random sets I_α for $\alpha = 1, 2, 3, 4$ are defined by

$$I_\alpha = \{1 \le i \le n : X_i = \alpha\}$$

(for the time being the I_α's do not satisfy the conditions $\#I_\alpha = n_\alpha$; they will be appropriately modified at the end of the proof).

Let $1 \le i \le m_1$, $1 \le j \le m_2$ and let z_{ij} be the (i,j)-th entry of the matrix $Q_1^*(p_\beta \Pi_{I_\alpha} Q_2 - p_\alpha \Pi_{I_\beta} Q_2)$. Clearly

$$z_{ij} = p_\beta \sum_{\{k : X_k = \alpha\}} x(k) y(k) - p_\alpha \sum_{\{k : X_k = \beta\}} x(k) y(k)$$

where $(x(1), \ldots, x(n))$ is the i-th column of Q_1 and $(y(1), \ldots, y(n))$ is the j-th column of Q_2.

Denote $M = \|Q_1\|_\infty \|Q_2\|_\infty$.

Claim. We have for $n > 1$

$$P[|z_{ij}| > 2M(n \log n)^{1/2}] < 2n^{-4}, \tag{19}$$

$$P[|\#I_\alpha - n_\alpha| > 2(n \log n)^{1/2}] < 2n^{-4}. \tag{20}$$

Indeed, let Y be a random variable such that

$$P(Y = p_\alpha) = p_\beta, P(Y = -p_\beta) = p_\alpha, P(Y = 0) = 1 - (p_\beta + p_\alpha),$$

let $Y_1, Y_2, \dots, Y_n$ be independent copies of Y. It is clear that z_{ij} is equidistributed with the random variable $S = \sum x(k)y(k)Y_k$. By Bernstein's inequality (cf. [4],1.3.2, p.12)

$$P(|S| > 2M(n \log n)^{1/2}) \leq 2n^{-4} \quad \text{for} \quad n > 1,$$

which is (19). (20) is obtained analogously taking Y such that $P(Y = 1) = p_\alpha, P(Y = 0) = 1 - p_\alpha$ and $S = Y_1 + \cdots + Y_n$. This proves the Claim.

Now we see that the probability that

$$|z_{ij}| \leq 2M(n \log n)^{1/2} \text{ for every } 1 \leq i \leq m_1, 1 \leq j \leq m_2 \tag{21}$$

and that also

$$|\#I_\alpha - p_\alpha| \leq 2(n \log n)^{1/2} \quad \text{for} \quad \alpha = 1, 2, 3, 4 \tag{22}$$

is greater than $1-(m_1 m_2+4)2n^{-4} \geq 1-32n^{-2}-8n^{-4}$. Thus for $n > 7$ there exist I_α's so that both (21) and (22) are satisfied. By (22) it is clear that by removing from or adding to I_α's fewer than $2(n \log n)^{1/2}$ elements, we can obtain disjoint sets so that $\#I_\alpha = n_\alpha$ for $\alpha = 1, 2, 3, 4$. This procedure will result in increasing $|z_{ij}|$ by at most $2M(n \log n)^{1/2}$. Consequently, for $n > 7$, the Lemma is true with $C = 4$. By adjusting C, it remains true for all $n > 1$. ■

The next lemma obviously implies (17) and this completes our proofs.

Lemma 5. *For $k = 0, 1, \dots$ there exist orthonormal systems $u_1^k, \dots, u_{3\cdot 2^k}^k$ in $\mathbb{R}^{3\cdot 2^k}$ such that, with the notation of* (16) *we have*

$$\|u_j^k\|_\infty \leq 2^{-\frac{k}{2}} \text{ for } \; j = 1, \dots, 3{\cdot}2^k; k = 0, 1, \dots \tag{23}$$

$$\|2(U_\varepsilon^k)^* U_\varepsilon^k - (V_\eta^k)^* V_\eta^k\|_\infty \leq Ck^2 2^{-\frac{k}{2}} \quad \text{for} \quad \varepsilon, \eta = 0, 1; k = 1, 2, \dots \tag{24}$$

$$\|(U_\varepsilon^{k+1})^* V_\varepsilon^k\|_\infty \leq Ck^2 2^{-\frac{k}{2}} \quad \text{for} \quad \varepsilon = 0, 1; k = 1, 2, \dots. \tag{25}$$

(observe that $(V_\varepsilon^{k-1})^* U_\varepsilon^k = [(U_\varepsilon^k)^* V^{k-1}\varepsilon]^*$, thus (25) gives also

$$\|(V_\varepsilon^{k-1})^* U_\varepsilon^k\|_\infty \leq Ck^2 2^{-\frac{k}{2}} \text{ for } \; \varepsilon = 0, 1; k = 2, 3, \dots.)$$

Proof. . Let $W^k = (u^k(i,j))_{1 \leq i,j \leq 3\cdot 2^k}$ for $k = 0, 1, 2, \dots$ be a $3{\cdot}2^k \times 3{\cdot}2^k$ orthogonal matrix with $\|W^k\|_\infty \leq 2^{-\frac{k}{2}}$ (e.g. we can take the matrix $\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$ tensored with the $2^k \times 2^k$ orthonormal Walsh matrix). We denote $v^k(i,j) = u^k(2^k + i, j)$ and define U^k, V^k by (15).

Let us first apply Lemma 4 to $Q_1 = Q_2 = W^k, k \geq 1$ with $n_1 = n_2 = 2^{k-1}, n_3 = n_4 = 2^k$. We obtain thus disjoint sets $I_0^k, I_1^k, J_0^k, J_1^k \subset \{1, \dots, 3{\cdot}2^k\}$ such that :

$$\#I_0^k = \#I_1^k = 2^{k-1}, \#J_0^k = \#J_1^k = 2^k \quad \text{and}$$

$$\|(W^k)^*(2\Pi_{I_\varepsilon^k} W^k - \Pi_{J_\eta^k} W^k)\|_\infty \leq Ck^2 2^{-\frac{k}{2}} \text{ for } \varepsilon, \eta = 0, 1. \tag{26}$$

By reordering the rows of W^k, we can assume that

$$I_0^k = \{1, \dots, 2^{k-1}\}, I_1^k = \{2^{k-1} + 1, \dots, 2^k\},$$
$$J_0^k = \{2^k + 1, \dots, 2^{k+1}\}, J_1^k = \{2^{k+1} + 1, \dots, 3{\cdot}2^k\}.$$

We see now that

$$(W^k)^*(2\Pi_{I_\varepsilon^k} W^k - \Pi_{J_\eta^k} W^k) =$$
$$=(U_\varepsilon^k + V_\eta^k)^*(2U_\varepsilon^k - V_\eta^k) = 2(U_\varepsilon^k)^* U_\varepsilon^k - (V_\eta^k)^* V_\eta^k,$$

thus (26) becomes (24).

To obtain (25), we apply Lemma 4 again: Let Q_1 be the $2^k \times 3{\cdot}2^k$ matrix consisting of the first 2^k rows of W^{k+1}, let Q_2 be the $2^k \times 3{\cdot}2^k$ matrix consisting of rows numbered $2^k + 1, 2^k + 2, \dots, 2^{k+1}$ of W^k. Applying Lemma 4 with $n_1 = n_2 = 2^{k-1}, n_3 = n_4 = 0$, we obtain a set $I \subset \{1, \dots, 2^k\}$ with $\#I = 2^{k-1}$ such that

$$\|Q_1^*(\Pi_{\{1,\dots,2^k\}\setminus I} Q_2 - \Pi_I Q_2)\|_\infty \leq Ck^2 2^{-\frac{k}{2}}. \tag{27}$$

Let us now modify W^{k+1} by multiplying the I-numbered rows of W^{k+1} by -1; the remaining rows are not changed. The modified matrix will still be called W^{k+1}. We see that then

$$Q_1^*(\Pi_{\{1,\dots,2^k\}\setminus I} Q_2 - \Pi_I Q_2) = (U_0^{k+1})^* V_0^k,$$

thus (27) becomes (25) for $\varepsilon = 0$. Analogously we obtain (25) for $\varepsilon = 1$. ■

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