Nine 20+ Year Old Problems in the Geometry of Banach Spaces

or:

(What I* have been doing this millenium)

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* **With a LOT of help!**

WBJ and E. Odell, *The diameter of the isomorphism class of a Banach space*, **Annals Math. 162** (2005), 423–437.

WBJ and G. Schechtman, *Very tight embeddings of subspaces of L_p, $1 < p < 2$, into ℓ_p^n*, **GAFA 13** (2003), 845–850.

..., *Multiplication operators on $L(L_p)$ and ℓ_p-strictly singular operators*

WBJ, B. Maurey, and G. Schechtman, *Weakly null sequences in L_1*, **JAMS 20** (2007), 25–36.

..., *Non-linear factorization of linear operators.*

V. P. Fonf, WBJ, G. Pisier, and D. Preiss, *Stochastic approximation properties in Banach spaces*, **Studia Math. 159** (2003), 103–119.

WBJ and A. Szankowski, *Complementably universal Banach spaces, II, and .....*

..., *Banach spaces all of whose subspaces have the approximation property, II.*

WBJ and B. Zheng, *A characterization of subspaces and quotients of reflexive Banach spaces with unconditional basis*, **Duke J. Math.**

WBJ and E. Odell, *The diameter of the isomorphism class of a Banach space.*

$D(X) = \sup\{d(X_1, X_2) : X_1, X_2$ are isomorphic to $X\}$.

[Schäffer, '76]: Is $D(X) = \infty$ for all infinite dimensional X?

The late V. Gurarii pointed out that $D(X) = \infty$ for many classes of spaces (super-reflexive; all "classical" spaces; ...). Indeed, if X is infinite dimensional and E is any finite dimensional space, them there is a space X_0 so that X is isomorphic to $E \oplus_2 X_0$. Therefore, if $D(X)$ is finite, then X is finitely complementably universal; that is, there is a constant C so that every finite dimensional space is C-isomorphic to a C-complemented subspace of X. This implies that X cannot have non trivial type or non trivial cotype or local unconditional structure or numerous other structures. In particular, X cannot be any of the classical spaces or be super-reflexive.

[J-O, '05]: The answer to Schäffer's question is yes for separable X.

Call a Banach space X K-*elastic* provided every isomorph of X K-embeds into X. Call X *elastic* if X is K-elastic for some $K < \infty$.

Theorem. *If X is a separable Banach space so that for some K, every isomorph of X is K-elastic, then X is finite dimensional.*

This of course implies that $D(X) = \infty$ if X is separable and infinite dimensional.

Conjecture. *If X is an elastic infinite dimensional separable Banach space, then $C[0,1]$ is isomorphic to a subspace of X.*

The proof of the Theorem could be considerably streamlined if the Conjecture has an affirmative answer.

Call a Banach space X K*-elastic* provided every isomorph of X K-embeds into X. Call X *elastic* if X is K-elastic for some $K < \infty$.

Theorem. *If X is a separable Banach space so that for some K, every isomorph of X is K-elastic, then X is finite dimensional.*

If X satisfies the hypothesis of the theorem but is infinite dimensional, then

1. c_0 must embed into X (uses Bourgain's index theory).

2. There must be a weakly null normalized sequence (WNNS) which has a spreading model that is neither ℓ_1 nor c_0.

The other two steps in the proof of the Theorem are of a general character. Both involve mathematics from the 1970s.

3. If the WNNS $\{x_n\}_{n=1}^{\infty}$ in a space Y has a spreading model that is neither ℓ_1 nor c_0, then $\forall C$, $\exists N$, there is a subsequence $\{y_n\}_{n=1}^{\infty}$ and an equivalent norm $|\cdot|$ on Y s.t. no subsequence of $\{y_n\}_{n=1}^{\infty}$ is block N-unconditional with constant C for the norm $|\cdot|$. (Uses [Mauey-Rosenthal, '77] considerations.)

A basic sequence $\{x_n\}_{n=1}^{\infty}$ *block N-unconditional with constant K* if every block basis $\{y_i\}_{i=1}^{N}$ of $\{x_n\}_{n=1}^{\infty}$ is K-unconditional; that is,

$$\|\sum_{i=1}^{N} \pm a_i y_i\| \leq K \|\sum_{i=1}^{N} a_i y_i\|$$

for all scalars $\{a_i\}_{i=1}^{N}$ and all choices of $\pm$.

4. If X is separable, then $\forall N$ there is an equivalent norm on X s.t. every WNNS has a subsequence which is block N-unconditional with constant 3 for the equivalent norm. (Uses [Lindenstrauss-Pełczyński, '71] considerations.)

WBJ and G. Schechtman, *Very tight embeddings of subspaces of L_p, $1 < p < 2$, into ℓ_p^n*, **GAFA 13** (2003), 845–850.

The first random embedding theorem for other than Hilbert spaces [J-S, '82] was that if $1 \leq p < r < 2$, then $\forall \epsilon > 0$ the space ℓ_r^n $1+\epsilon$–embeds into ℓ_p^N with $N = K(p,r,\epsilon)n$. When $r = 2$, this comes out of Milman's approach to Dvoretzky theorem [Figiel-Lindenstrauss-Milman, '77]. But another approach [Kashin, '77] also yielded that for $r = 2$, the space ℓ_r^n K–embeds into $\ell_p^{(1+\epsilon)n}$ with $K = K(\epsilon)$. Whether Kashin's theorem was valid for $p < r < 2$, left open in [J-S, '82], was proved in [J-S, '03]. The main new tool was a theorem in [Bourgain-Kalton-Tzafriri, '89] that says (qualitatively speaking) that if Q is a quotient map from ℓ_p^n onto a space with dimension proportional to n, then the restriction of Q to some proportional dimensional coordinate subspace is a good isomorphism. By iterating this theorem, [J-S, '03] show that ℓ_r^n K–embeds into $\ell_p^{(1+\epsilon)n}$ with $K = K(p,r,\epsilon)$.

The [J-S, '82] result was extended and generalized by [Pisier, '83], [S, '87], [Bourgain-Lindenstrauss-Milman, '89], [Talagrand, '90]. These works and [J-S, '03] yield that every n dimensional subspace of L_r must K–embed into $\ell_p^{(1+\epsilon)n}$ with $K = K(p, r, \epsilon)$ when $1 \leq p < r < 2$.

One open problem is whether there is a purely random way of embedding ℓ_r^n into $\ell_p^{(1+\epsilon)n}$. Before [J-S, '03], only random tools were used [Naor-Zvavitch, '01] to show that ℓ_r^n K–embeds into $\ell_1^{(1+\epsilon)n}$ with $K = K(r, \epsilon)(\log n)^{f(r,\epsilon)}$.

WBJ, B. Maurey, and G. Schechtman, *Weakly null sequences in L_1*, **JAMS 20** (2007), 25–36.

The first weakly null normalized sequences (WNNS) with no unconditional subsequence were constructed in [Maurey-Rosenthal, '77]. Their technique was incorporated into [Gowers-Maurey, '92], but the examples in [M-R, '77] are still interesting because the ambient spaces were $C(K)$ with K countable, which are hereditarily c_0. Every subsequence of the WNNS they constructed reproduces the (conditional) summing basis on blocks.

[Maurey-Rosentha], '77] asked whether every WNNS sequence in L_1 has an unconditional subsequence. In [J-M-S, '07] we construct a WNNS in L_1 s.t. every subsequence contains a block basis that is $1+\epsilon$–equivalent to the (conditional) Haar basis for L_1. In fact, the theorem stated this way extends to rearrangement invariant spaces which (in some appropriate sense) are not to the right of L_2 and which are not too close to L_∞.

WBJ, B. Maurey, and G. Schechtman, *Non-linear factorization of linear operators,* (submitted).

[J-M-S, 07?] gives an affirmative answer to the question [Heinrich-Mankiewicz, '82] whether the $\mathcal{L}_1$ spaces are preserved under uniform equivalence (i.e., f, f^{-1} both uniformly continuous).

At the heart of the question is a recurring problem:

Suppose a linear mapping $T : X \to Y$ admits a Lipschitz factorization through a Banach space Z; i.e., we have Lipschitz $F_1 : X \to Z$ and $F_2 : Z \to Y$ and $F_2 \circ F_1 = T$. What extra is need to guarantee that T admits a linear factorization through Z?

Something extra is needed because the identity on $C[0,1]$ Lipschitz factors through c_0 [Aharhoni, '74], [Lindenstrauss, '64].

The main result in [J-M-S, 07?] is

Theorem. *Let X be a finite dimensional normed space, Y a Banach space with the RNP and $T : X \to Y$ a linear operator. Let Z be a separable Banach space and assume there are Lipschitz maps $F_1 : X \to Z$ and $F_2 : Z \to Y$ with $F_2 \circ F_1 = T$. Then for every $\lambda > 1$ there are linear maps $T_1 : X \to L_\infty(Z)$ and $T_2 : L_1(Z) \to Y$ with $T_2 \circ i_{\infty,1} \circ T_1 = T$ and $\|T_1\| \cdot \|T_2\| \leq \lambda \mathrm{Lip}\,(F_1)\mathrm{Lip}\,(F_2)$.*

Notice that if Z is $\mathcal{L}_1$ then so is $L_1(Z)$ and hence T linearly factors through a $\mathcal{L}_1$ space. This and fairly standard tools in non linear geometric functional analysis give an affirmative answer to the problem from [H-M, '82].

The proof of the Theorem is based on a rather simple local-global linearization idea. For the application we need only the case where Y is finite dimensional.

V. P. Fonf, WBJ, G. Pisier, and D. Preiss, *Stochastic approximation properties in Banach spaces,* **Studia Math. 159** (2003), 103–119.

A Banach space X has the *stochastic approximation property* (SAP) provided for every Radon measure μ on X there is a sequence (T_n) of finite rank operators on X s.t. for μ-a.e. $x \in X$, $T_n x \to x$. If The T_n can always be chosen to be the partial sum projections $T_n = \sum_{i=1}^n x_i^* \otimes x_i$ associated associated with a fundamental and total biorthogonal system (x_n, x_n^*), we say that X has the *stochastic basis property* (SBP).

[Kwapien-Rosinski, '80] asked whether every Banach space has the stochastic approximation property.

Theorem. *[F-J-P-P, '03] If X has non trivial type and has the SAP, then X has the approximation property.*

However, the SBP may still be useful for for probability in Banach spaces because it is proved in [F-J-P-P, '03] that the SAP and the SBP are equivalent.

WBJ and A. Szankowski, *Complementably universal Banach spaces, II, and .....*

Given a family $\mathcal{F}$ of operators between Banach spaces, it is natural to try to find a single (usually separable) Banach space Z s.t. all the operators in $\mathcal{F}$ factor through Z. If $\mathcal{F}$ is the collection of all operators between separable Banach spaces that have the bounded approximation property, there is such a separable Z; namely, the separable universal basis space of [Pełczyński, '69], [Kadec, '71], [Pełczyński, '71]. This space, as well as smaller (even reflexive) spaces [J, '71] have the property that every operator that is uniformly approximable by finite rank operators factor through Z. Now there is not a separable space s.t. every operator between separable spaces factors through it [J-Szankowski, '76], but this paper left open the possibility that there is a separable space s.t. every *compact* operator factors through it. It turns out that no such space exists [J-S, 08?].

WBJ and A. Szankowski, *Banach spaces all of whose subspaces have the approximation property, II.*

A Banach space has the *hereditary approximation property* (HAP) provided every subspace has the approximation property. There are non Hilbertian spaces that have the HAP [J, '80], [Pisier, '88]. All of these examples are *asymptotically Hilbertian*; i.e., for some K and every n, there is a finite codimensional subspace all of whose n-dimensional subspaces are K-isomorphic to ℓ_2^n. An asymptotically Hilbertian space must be superreflexive and cannot have a symmetric basis unless it is isomorphic to a Hilbert space. This led to two problems [J, '80]:

1. Can a non reflexive space have the HAP?

2. Does there exist a non Hilbertian space with a symmetric basis which has the HAP?

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2. Does there exist a non Hilbertian space with a symmetric basis which has the HAP?

The HAP is very difficult to work with. It does not have good permanence properties– there are spaces X and Y which have the HAP s.t. $X \oplus Y$ fails the HAP [Casazza-Garcia-J, '01].

The main result of [J-Szankowski2, '08?] gives an affirmative answer to problem 2 from [J, '80]:

Theorem. *There is a function $f(n) \uparrow \infty$ s.t. if for infinitely many n we have $D_n(X) \leq f(n)$, then X has the HAP.*

Here $D_n(X) := \sup d(E, \ell_2^n)$, where the sup is over all n-dimensional subspaces of X. The proof combines the ideas in [J, '80] with the argument in [Lindenstrauss-Tzafriri, '76].

The estimate of $f(n)$ is good enough to show that $\ell_2(X)$ has the HAP for every weak Hilbert space X that has an unconditional basis, but it remains open whether $\ell_2(X)$ has the HAP for every weak Hilbert space X.

WBJ and B. Zheng, *A characterization of subspaces and quotients of reflexive Banach spaces with unconditional basis*.

A problem that goes back to the 1970s is to give an intrinsic characterization of Banach spaces that embed into a space that has an unconditional basis. It was shown that every space with an unconditional expansion of the identity (in particular, every space with an unconditional finite dimensional decompostion) embeds into a space with unconditional basis [Pełczyński-Wojtaszczyk, '71], [Lindenstrauss-Tzafriri, '77].

The only apparent useful invariant is that in a subspace of a space with unconditional basis, every WNNS has an unconditional basic sequence. A quotient of a space with shrinking unconditional basis has this property [J, '77], [Odell, '86]. Also, a reflexive quotient X of a space with shrinking unconditional basis embeds into a space with unconditional basis as long as X has the approximation property [Feder, '80].

So there were two problems

1. Give an intrinsic characterization of Banach spaces that embed into a space that has an unconditional basis.

2. Does every quotient of a space with shrinking unconditional basis embed into a space with unconditional basis?

Much research centered around reflexive spaces. Every reflexive subspace of a space with unconditional basis embeds into a reflexive space with unconditional basis [Davis-Figiel-J-Pełczyński, '74], [Figiel-J-Tzafriri, '75].

In [J-Zheng, 2007?] both problems are given affirmative answers for reflexive spaces (and since they have given an affirmative answer to (2) in general). The answers follow from the following theorem:

Theorem. *Let X be a separable reflexive Banach space. Then the following are equivalent.*

X has the UTP.

X is isomorphic to a subspace of a Banach space with an unconditional basis.

X is isomorphic to a subspace of a reflexive space with an unconditional basis.

X is isomorphic to a quotient of a Banach space with a shrinking unconditional basis.

X is isomorphic to a quotient of a reflexive space with an unconditional basis.

X is isomorphic to a subspace of a quotient of a reflexive space with an unconditional basis.

X is isomorphic to a subspace of a reflexive quotient of a Banach space with a shrinking unconditional basis.

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X is isomorphic to a quotient of a reflexive subspace of a Banach space with a shrinking unconditional basis.

X^* has the UTP.

Definition. *[Odell-Schlumprecht] A branch of a tree is a maximal linearly ordered subset of the tree under the tree order. We say X has the C-unconditional tree property (C-UTP) if every normalized weakly null infinitely branching tree in X has a C-unconditional branch. X has the UTP if X has the C-UTP for some $C > 0$.*

The UTP is a strengthening of the property "every WNNS has an unconditional subsequence". The weaker property for a reflexive space does NOT imply embeddability into a space with unconditional basis [J-Zh, '07?].

The proof of the theorem uses some new tricks, blocking methods developed in the 1970s [J-Zippin, '72, '74], [J-Odell, '74, 81], [J, '77], and the analysis in [Odell-Schlumprecht, '02,'06] relating tree properties to embeddability into spaces that have a finite dimensional decomposition with the corresponding skipped blocking property.

WBJ and G. Schechtman, *Multiplication operators on $L(L_p)$ and ℓ_p-strictly singular operators*

The main positive result in this paper is that if T is an ℓ_p-strictly singular operator on L_p, $1 < p < 2$, and $T_{|X}$ is an isomorphism, then X embeds into L_r for all $r < 2$. We also give an example of such a T and X s.t. X is not isomorphic to a Hilbert space. At first we thought that T_ϵ, convolution by an ϵ-biased coin on L_p of the Cantor group $\Delta := \{-1, 1\}^{\mathbb{N}}$ might be a counterexample for some p and ϵ, but it turned out that T_ϵ can be an isomorphism only on Hilbertian subspaces of L_p when $p > 1$. Our argument allowed us to prove an analogous theorem for $L_1(G)$ even though T_ϵ is NOT ℓ_1-SS on L_1. Namely, if T_ϵ is an isomorphism on a REFLEXIVE subspace X of $L_1(G)$, then X is isomorphic to a Hilbert space. This answered a question asked by Rosenthal in 1978.