LIPSCHITZ p-SUMMING OPERATORS

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ABSTRACT. The notion of Lipschitz p-summing operator is introduced. A non linear Pietsch factorization theorem is proved for such operators and it is shown that a Lipschitz p-summing operator that is linear is a p-summing operator in the usual sense.

1. INTRODUCTION

In this note we introduce a natural non linear version of p-summing operator, which we call Lipschitz p-summing operator. In section 2 we prove a non linear version of the Pietsch factorization theorem, show by example that the strong form of the Pietsch domination theorem is not true for Lipschitz p-summing operators, and make a few other remarks about these operators. In section 3 we "justify" our nomenclature by proving that for a linear operator, the Lipschitz p-summing norm is the same as the usual p-summing norm. Finally, in section 4 we raise some problems which we think are interesting.

2. PIETSCH FACTORIZATION

The *Lipschitz p-summing ($1 \le p < \infty$) norm*, $\pi_p^L(T)$, of a (possibly non linear) mapping $T\colon X \to Y$ between metric spaces is the smallest constant C so that for all (x_i), (y_i) in X and all positive reals a_i

$$\sum a_i \|Tx_i - Ty_i\|^p \le C^p \sup_{f \in B_{X^\#}} \sum a_i |f(x_i) - f(y_i)|^p \tag{2.1}$$

Here $B_{X^\#}$ is the unit ball of $X^\#$, the Lipschitz dual of X, i.e., $X^\#$ is the space of all real valued Lipschitz functions under the (semi)-norm $\mathrm{Lip}(\cdot)$; and $\|x-y\|$ is the distance from x to y in Y. We follow the usual convention of considering X as a pointed metric space by designating a special point $0 \in X$ and identifying $X^\#$ with the Lipschitz functions on X that are zero at 0. With this convention $(X^\#, \mathrm{Lip}(\cdot))$ is a Banach space and $B_{X^\#}$ is a compact Hausdorff space in the topology of pointwise convergence on X.

Notice that the definition is the same if we restrict to $a_i = 1$. Indeed, by approximation it is enough to consider rational a_i and thus, by clearing denominators, integer a_i. Then, given a_i, x_i, and y_i, consider the new collection of vectors in which the pair (x_i, y_i) is repeated a_i times. (This observation was made with M. Mendel and G. Schechtman.)

It is clear that π_p^L has the ideal property; i.e., $\pi_p^L(ATB) \le \mathrm{Lip}(A)\pi_p^L(T)\mathrm{Lip}(B)$ whenever the compositions make sense. Also, if Y is a Banach space, the space of

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Lipschitz p-summing maps from any metric space into Y is a Banach space under the norm π_p^L.

If T is a linear operator, it is clear that $\pi_p^L(T) \le \pi_p(T)$, where $\pi_p(\cdot)$ is the usual p-summing norm [5, p. 31]. In section 3 we prove that the reverse inequality is true.

We begin with a Pietsch factorization theorem for Lipschitz p-summing operators.

Theorem 1. *The following are equivalent for a mapping $T\colon\ X \to Y$ between metric spaces and $C \ge 0$.*

(1) $\pi_p^L(T) \le C$.

(2) *There is a probability μ on $B_{X^\#}$ such that*

$$\|Tx - Ty\|^p \le C^p \int_{B_{X^\#}} |f(x) - f(y)|^p \, d\mu(f).$$

(Pietsch domination.)

(3) *For some (or any) isometric embedding J of Y into a 1-injective space Z, there is a factorization*

$$\begin{array}{ccccc} L_\infty(\mu) & & \xrightarrow{\quad I_{\infty,p}\quad} & & L_p(\mu) \\ A\uparrow & & & & \downarrow B \\ X & \xrightarrow{T} & Y & \xrightarrow{\quad J\quad} & Z \end{array}$$

with μ a probability and $\mathrm{Lip}(A) \cdot \mathrm{Lip}(B) \le C$.

(Pietsch factorization.)

Proof. That (2) implies (3) is basically obvious: Let $A\colon\ X \to L_\infty(\mu)$ be the natural isometric embedding composed with the formal identity from $C(B_{X^\#})$ into $L_\infty(\mu)$. Then (2) says that the Lipschitz norm of B restricted to $I_{\infty,p}AX$ is bounded by C, which is just (3). (We have used implicitly the well known fact that every metric space embeds into $\ell_\infty(\Gamma)$ for some set Γ and that, by the non linear Hahn-Banach theorem, $\ell_\infty(\Gamma)$ is 1-injective. See Lemma 1.1 in [3].)

For (3) implies (1), use

$$\begin{aligned} \pi_p^L(T) &= \pi_p^L(JT) \le \mathrm{Lip}(A)\pi_p^L(I_{\infty,p})\mathrm{Lip}(B) \le \mathrm{Lip}(A)\pi_p(I_{\infty,p})\mathrm{Lip}(B) \\ &= \mathrm{Lip}(A)\mathrm{Lip}(B). \end{aligned}$$

The proof of the main implication, that (1) implies (2), is like the proof of the (linear) Pietsch factorization theorem (see, e.g., [5, p. 44]). Suppose $\pi_p^L(T) = 1$. Let Q be the convex cone in $C(B_{X^\#})$ consisting of all positive linear combinations of functions of the form $\|Tx - Ty\| - C^p|f(x) - f(y)|^p$, as x and y range over X. Condition (1) says that Q is disjoint from the the positive cone $P = \{F \in C(B_{X^\#}) \mid F(f) > 0 \; \forall f \in X^\#\}$, which is an open convex subset of $C(B_{X^\#})$. Thus by the separation theorem and the Riesz representation theorem there is a finite signed Baire measure μ on $B_{X^\#}$ and a real number c so that for all $G \in Q$ and $F \in P$, $\int_{X^\#} G \, d\mu \le c < \int_{X^\#} F \, d\mu$. Since $0 \in Q$ and all positive constants are in P, we see that $c = 0$, and since $\int_{X^\#} \cdot \, d\mu$ is positive on the positive cone P of $C(B_{X^\#})$, the signed measure μ is a positive measure, which we can assume by rescaling is a probability measure. It is clear that the inequality in (2) is satisfied. ☐

It is worth noting that the conditions in Theorem 1 are also equivalent to

(4) *There is a probability μ on K, the closure in the topology of pointwise convergence on X of the extreme points of $B_{X^\#}$, so that*

$$\|Tx - Ty\|^p \le C^p \int_K |f(x) - f(y)|^p \, d\mu(f).$$

The proof that (1) implies (4) is the same as the proof that (1) implies (2) since the supremum on the right side of (2.1), the definition of the Lipschitz p-summing norm, is the same as

$$\sup_{f \in K} \sum a_i |f(x_i) - f(y_i)|^p.$$

One immediate consequence of Theorem 1 is that $\pi_p^L(T)$ is a monotonely decreasing function of p. Another consequence is that there is a version of Grothendieck's theorem (that every linear operator from an L_1 space to a Hilbert space is 1-absolutely summing). In the category of metric spaces with Lipschitz mappings as morphisms, weighted trees play a role analogous to that of L_1 in the linear theory. In particular, every finite weighted tree has the lifting property, which is to say that if X is a finite weighted tree, $T\colon\ X \to Y$ is a Lipschitz mapping from X into a metric space Y, and $Q\colon\ Z \to Y$ is a 1-Lipschitz quotient mapping in the sense of [2], [7], then for each $\varepsilon > 0$ there is a mapping $S\colon\ X \to Z$ so that $\mathrm{Lip}(S) \le \mathrm{Lip}(T) + \varepsilon$ and $T = QS$. Letting Y be a Hilbert space and Z an L_1 space, we see from Grothendieck's theorem and the ideal property of π_1^L that if every finite subset of X is a weighted tree (in particular, if X is a tree or a metric tree–see [7]), then $\pi_1^L(T) \le K_G \mathrm{Lip}(T)$, where K_G is Grothendieck's constant. Here we use the obvious fact that $\pi_p^L(T\colon\ X \to Y)$ is the supremum of $\pi_p^L(T_{|K})$ as K ranges over finite subsets of X.

The strong form of the Pietsch domination theorem says that if X is a subspace of $C(K)$ for some compact Hausdorff space K, and T is a p-summing linear operator with domain X, then there is a probability measure μ on K so that for all $x \in X$, $\|Tx\|^p \le \pi_p(T)^p \int_K |x(t)|^p \, d\mu(t)$. It is easy to see that there is not a non linear version of this result. Let D_n be the discrete metric space with n points so that the distance between any two distinct points is one. We can embed D_n into $C(\{-1,1\}^n)$ in two essentially different ways. First, if $D_n = \{x_1, \dots, x_n\}$, let $f(x_k) = \frac{1}{2} r_k$, where r_k is the projection onto the kth coordinate. The image of this set under the canonical injection from $C(\{-1,1\}^n)$ into $L_p(\{-1,1\}^n, \mu)$ with μ the uniform probability on $\{-1,1\}^n$ is a discrete set with the p-th power of all distances one-half. This shows that the identity on D_n has Lipschitz p-summing norm at most two. Secondly, let $g(k)$, $1 \le k \le n$, be disjointly supported unit vectors in $C(\{-1,1\}^n)$. Then for any probability measure ν on $\{-1,1\}^n$, the injection from $C(\{-1,1\}^n)$ into $L_p(\{-1,1\}^n, \nu)$ shrinks the distance between some pair of the $g(k)$'s to at most $(2/n)^{1/p}$.

Incidentally, $\pi_p^L(I_{D_n})$ tends to $2^{\frac{1}{p}}$ as $n \to \infty$ and can be computed exactly. To see this, note that the extreme points, K_n, of $B_{D_n^\#}$ are of the form $\pm\chi_A$ with A a non empty subset of $D_n \sim \{0\}$. This can be calculated directly or deduced from Theorem 1 in [6]. We calculate $\pi_p^L(I_{D_n})$ in the (easier) case that n is even. Define a probability μ on K_n by letting μ be the uniform measure on $J_{n/2} := \{\chi_A\colon\ |A| = n/2, A \subset D_n \sim \{0\}\}$ (so that $\mu(e) = 0$ for elements e of $K_n \sim J_{n/2}$). Then for each pair of distinct points x and y in D_n, $\int_{K_n} |f(x) - f(y)|^p \, d\mu(f) = \frac{n}{2(n-1)}$, so that $\pi_p^L(I_{D_n}) \le (2 - \frac{2}{n})^{\frac{1}{p}}$. To see that μ is a Pietsch measure for I_{D_n}, let

ν be any Pietsch probability for I_{D_n} on K_n. We can clearly assume that ν is supported on the positive elements in K_n. By averaging ν against the permutations of D_n which fix 0, which is a group of isometries on D_n, we get another Pietsch probability for I_{D_n} (which we continue to denote by ν) so that if we condition ν on $J_k := \{\chi_A\colon\ |A| = k, A \subset D_n \sim \{0\}\}$, $1 \leq k \leq n-1$, the resulting probability ν_k on J_k is the uniform probability. A trivial calculation shows that for x, y in $D_n \sim \{0\}$, $\int_{J_k} |f(x) - f(y)|^p \, d\nu_k(f) \leq \frac{n}{2(n-1)}$. This proves that μ is a Pietsch measure for I_{D_n} and hence $\pi_p^L(I_{D_n}) = (2 - \frac{2}{n})^{\frac{1}{p}}$.

Our final comment on Lipschitz 1-summing operators is that the concept has appeared in the literature even if the definition is new. In [4], Bourgain proved that every n point metric space can be embedded into a Hilbert space with distortion at most $C \log n$, where C is an absolute constant. In fact, he really proved the much stronger result that $\pi_1^L(I_X) \leq C \log n$ if I_X is the identity mapping on an n point space X by making use of a special embedding of X into a space $C(K_X)$ with K_X a finite metric space and constructing a probability on K_X. Moreover, Bourgain's construction has occasionally been used in the computer science literature. The strong form of Bourgain's theorem is also used in [8] to prove an inequality that is valid for all metric spaces.

3. Linear operators

In this section we show that the Lipschitz p-summing norm of a linear operator is the same as its p-summing norm. This justifies that the notion of Lipschitz p-summing operator is really a generalization of the concept of linear p-summing operator.

Theorem 2. *Let u be a bounded linear operator from X into Y and $1 \leq p < \infty$. Then $\pi_p^L(u) = \pi_p(u)$.*

Proof. Note that we can assume, without loss of generality, that $\dim Y \leq \dim X = N < \infty$. Indeed, it is clear from the definition that $\pi_p^L(u)$ is the supremum of $\pi_p^L(u_{|E})$ as E ranges over finite dimensional subspaces of X and similarly for $\pi_p^L(u)$. That we can assume $\dim Y \leq \dim X$ is clear from the linearity of u.

Since $\dim Y \leq N$, there is an embedding J of Y into ℓ_∞^m with $m \leq (\frac{3}{\varepsilon})^N$ so that $\|J\| = 1$ and $\|J^{-1}\| \leq 1 + \varepsilon$. We then get the following non linear Pietsch factorization:

$$\begin{array}{ccccc} L_\infty(\mu) & & \xrightarrow{i_{\infty,p}} & & L_p(\mu) \\ \alpha \uparrow & & & & \downarrow \beta \\ X & \xrightarrow{u} & Y & \xrightarrow{J} & \ell_\infty^m \end{array}$$

where $\mathrm{Lip}(\alpha) = 1$, $\mathrm{Lip}(\beta) \leq \pi_p^L(Ju) \leq \pi_p^L(u)$. We can also assume, without loss of generality, that the probability μ is a separable measure.

We now use some non linear theory that can be found in the book [3].

(1) The mapping α is weak* differentiable almost everywhere. This means that for (Lebesgue) almost every x_0 in X, there is a linear operator $D_{x_0}^{w^*}(\alpha)\colon\ X \to L_\infty(\mu)$ so that for all $f \in L_1(\mu)$ and for every $y \in X$,

$$\lim_{t \to 0} \left\langle \frac{\alpha(x_0 + ty) - \alpha(x_0)}{t}, f \right\rangle = \langle D_{x_0}^{w^*}(\alpha)(y), f \rangle.$$

(2) The operator $i_{\infty,p}\alpha$ is differentiable almost everywhere. This means that for almost every x_0 in X, there is a linear operator $D_{x_0}(i_{\infty,p}\alpha)\colon X \to L_p(\mu)$ so that

$$\sup_{\|y\|\le 1}\left\|\frac{i_{\infty,p}\alpha(x_0+ty)-i_{\infty,p}\alpha(x_0)}{t}-D_{x_0}(i_{\infty,p}\alpha)(y)\right\|_p \to 0 \quad \text{as} \quad t\to 0.$$

When $1<p<\infty$, statement (2) follows from the reflexivity of L_p (see [3, Corollary 5.12 & Proposition 6.1]). For $p=1$, just use (2) for $p=2$ and compose with $i_{2,1}$.

The mapping $i_{\infty,p}$ is weak* to weak continuous, so $D_{x_0}(i_{\infty,p}\alpha) = i_{\infty,p}D^{w^*}_{x_0}(\alpha)$ whenever both derivatives exist. Since they both exist almost everywhere, by making several translations we can assume without loss of generality that this equation is true for $x_0 = 0$ and also that $\alpha(0)=0$.

Next we show that in the factorization diagram the non linear map α can be replace by the linear operator $D^{w^*}_0(\alpha)$ by constructing a mapping $\tilde\beta\colon\ L_p(\mu)\to \ell^m_\infty$ so that $\tilde\beta i_{\infty,p}D^{w^*}_0(\alpha) = Ju$ and $\mathrm{Lip}(\tilde\beta)\le \mathrm{Lip}(\beta)$. To do this, define $\beta_n\colon\ L_p(\mu)\to\ell^m_\infty$ by $\beta_n(y) := n\beta(\frac{y}{n})$ and note that $\mathrm{Lip}(\beta_n)=\mathrm{Lip}(\beta)$. We have for each x in X

$$\begin{aligned}\|Ju(x)-\beta_n i_{\infty,p}D^{w^*}_0(\alpha)(x)\| &= \|\beta_n n i_{\infty,p}\alpha(x/n)-\beta_n D_0(i_{\infty,p}\alpha)(x)\|\\ &\le \mathrm{Lip}(\beta)\|n i_{\infty,p}\alpha(x/n)-D_0(i_{\infty,p}\alpha)(x)\|\end{aligned}$$

which tends to zero as $n\to\infty$. For $\tilde\beta$ we can take any cluster point of β_n in the space of functions from $L_p(\mu)$ into ℓ^m_∞; such exist because β_n is uniformly Lipschitz and $\beta_n(0)=0$.

Summarizing, we see that we have a factorization

$$\begin{array}{ccccc} L_\infty(\mu) & & \xrightarrow{i_{\infty,p}} & & L_p(\mu)\\ \tilde\alpha\uparrow & & & & \downarrow\tilde\beta\\ X & \xrightarrow{u} & Y & \xrightarrow{J} & \ell^m_\infty \end{array}$$

with $\tilde\alpha$ linear, $\|\tilde\alpha\|\le \mathrm{Lip}(\alpha)$, and $\mathrm{Lip}(\tilde\beta)\le\mathrm{Lip}(\beta)$.

The final step involves replacing $\tilde\beta$ with a linear operator. Since the restriction $\overline{\beta}$ of $\tilde\beta$ to the linear subspace $i_{\infty,p}\tilde\alpha[X]$ is linear and ℓ^m_∞ is reflexive, this follows from [3, Theorem 7.2], which is proved by a simple invariant means argument. Alternatively, one can use the injectivity of ℓ^m_∞ to extend $\overline{\beta}$ to $L_p(\mu)$. □

4. Open problems and concluding remarks

Problem 1. *Is there a composition formula for Lipschitz p-summing operators? That is, do we have $\pi^L_p(TS)\le\pi^L_r(T)\pi^L_s(S)$, when $\frac{1}{p}\le(\frac{1}{r}+\frac{1}{s})\wedge 1$?*

Say that a Lipschitz mapping $T\colon\ X\to Y$ is Lipschitz p-integral if it satisfies a factorization diagram as in condition (3) of Theorem 1, except with J being the canonical isometry from Y into $(Y^{\#})^*$. We then define the Lipschitz p-integral norm $I^L_p(T)$ of T to be the infimum of $\mathrm{Lip}(A)\cdot\mathrm{Lip}(B)$, the infimum being taken over all such factorizations. When T is a linear operator, this is the same as the usual p-integral norm of T. Indeed, in this case one can use for J the canonical isometry from Y into Y^{**} because Y^{**} is norm one complemented in $(Y^{\#})^*$. Then the proof that $I_p(T)\le I^L_p(T)$ is identical to the proof of Theorem 2.

Problem 2. *Is every Lipschitz 2-summing operator Lipschitz 2-integral?*

In the case where the target space Y is a Hilbert space, problem 2 has an affirmative answer by Kirszbraun's theorem [3, p. 18]. If Y has K. Ball's Markov cotype 2 property [1], it follows from Ball's work that the answer is still positive, although his result does not yield that $I_p^L(T)$ and $\pi_p^L(T)$ are equal. It is worth mentioning that the work of Naor, Peres, Schramm, and Sheffield [9] combines with Ball's result to yield that for $2 \le p < \infty$, every Lipschitz p-summing operator into L_r, $1 < r \le 2$, is Lipschitz p-integral.

We mentioned in section 2 that $\Pi_p^L(X,Y)$, the class of Lipschitz p-summing operators from X into Y, is a Banach space under the norm $\pi_p^L(\cdot)$ when Y is a Banach space.

Problem 3. *When Y is a Banach space and X is finite, what is the dual of $\Pi_p^L(X,Y)$?*

In section 2 we noted that there is a version of Grothendieck's theorem that is true in the non linear setting. Are there other versions? In particular, we ask the following.

Problem 4. *Is every Lipschitz mapping from an L_1 space to a Hilbert space Lipschitz 1-summing? Is every Lipschitz mapping from a $C(K)$ space to a Hilbert space Lipschitz 2-summing?*

It is elementary that for a linear operator $T\colon X \to Y$, $\pi_p(T)$ is the supremum of $\pi_p(TS)$ as S ranges over all operators from $\ell_{p'}$ into X of norm at most one. This leads us to ask

Problem 5. *If $T\colon X \to Y$ is Lipschitz, is $\pi_p^L(T)$ is the supremum of $\pi_p^L(TS)$ as S ranges over all mappings from finite subsets of $\ell_{p'}$ into X having Lipschitz constant at most one?*

Since all finite metric spaces embed isometrically into ℓ_∞, the answer to problem 5 is yes for $p = 1$.

Of course, all of the above problems are special cases of the general

Problem 6. *What results about p-summing operators have analogues for Lipschitz p-summing operators?*

Added in proof. Problem 3 has been solved by J. A. Chávex Domínguez (unpublished).

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