THE TRACE FORMULA IN BANACH SPACES

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Abstract. A classical result of Grothendieck and Lidskii says that the trace formula (that the trace of a nuclear operator is the sum of its eigenvalures provided the sequence of eigenvalues is absolutely summable) holds in Hilbert spaces. In 1988 Pisier proved that weak Hilbert spaces satisfy the trace formula. We exhibit a much larger class of Banach spaces, called Γ-spaces, that satisfy the trace formula. A natural class of asymptotically Hilbertian spaces, including some spaces that are ℓ_2 sums of finite dimensional spaces, are Γ-spaces. One consequence is that the direct sum of two Γ-spaces need not be a Γ-space.

Dedicated to the memory of Joram Lindenstrauss

1. Introduction

Let X be a Banach space. $L(X)$ denotes the space of bounded operators on the space X while $F(X)$ denotes the finite rank operators in $L(X)$. B_X denotes the unit ball of X. The identity operator on X is written I_X.

For $x^* \in X^*, x \in X$ let $x^* \otimes x \in F(X)$ be defined by

$$(x^* \otimes x)(y) = x^*(y)x.$$

Every $T \in F(X)$ can be represented in the form

$$T = \sum_{i=1}^{n} x_i^* \otimes x_i \text{ with } x_1^*, \dots, x_n^* \in X^*, x_1, \dots, x_n \in X.$$

By elementary algebra, the sum

$$\text{tr } T = \sum_{i=1}^{n} x_i^*(x_i)$$

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is well defined, i.e. does not depend on the representation of T. A much deeper elementary fact is that for $T \in F(X)$ the *trace formula*:

$$\text{tr } T = \sum \lambda_j(T) \tag{1}$$

holds. Here $\lambda_1(T), \lambda_2(T), \dots$ are all the eigenvalues of T, with their multiplicities (we suppose that X is a complex Banach space).

It is natural to seek generalizations of these facts to the infinite dimensional setting.

A $T \in B(X)$ is called *nuclear* if

$$T = \sum_{i=1}^{\infty} x_i^* \otimes x_i \text{ with } \sum \|x_i^*\| \|x_i\| < \infty.$$

By $N(T)$ we denote the space of all nuclear operators on X. In $N(X)$ we define the following norm (called the *nuclear norm*):

$$\|T\|_\wedge = \inf\{\sum \|x_i^*\| \|x_i\| : T = \sum_{i=1}^{\infty} x_i^* \otimes x_i\}. \tag{2}$$

Grothendieck [4] (cf. [11, Theorem 1.a.4.]) discovered that if X has the approximation property (AP), then for every $T \in N(X)$, tr $T = \sum_{i=1}^{\infty} x_i^*(x_i)$ is well defined.

Suppose that X is a complex Banach space with the AP. We ask whether

(L) the trace formula (1) holds for every $T \in N(X)$ which has absolutely summable eigenvalues $\lambda_1(T), \lambda_2(T), \dots$

(this assumption is necessary, because for every X not isomorphic to a Hilbert space there is a $T \in N(X)$ such that $\sum |\lambda_j(T)| = \infty$, by a result in [7]).

In [9] Lidskii proved that the answer to (L) is positive if X is a Hilbert space. As was pointed out by Pisier [15], Grothendieck was aware of this result somewhat earlier [5].

For general X, the answer is negative. Spaces which satisfy condition (L) will be called *Lidskii spaces*.

It turns out that Lidskii spaces are very close to Hilbert spaces. Let us say that X is *an almost Hilbert space* if X is of type $(2-\varepsilon)$ and of cotype $(2+\varepsilon)$ for every $\varepsilon > 0$.

Recall the definitions of type and cotype. Let (r_i) be a sequence of independent random variables taking the values 1 and -1 each with probability 1/2. Given $n \in \mathbb{N}$ and $1 \le p < \infty$, the type p constant $T_p^{(n)}$ and the cotype p constant $C_p^{(n)}$ are the smallest constants which satisfy the following inequalities for all n-tuples of vectors in X:

$$\mathbb{E}\|\sum_{i=1}^{n} r_i y_i\|^p \leq T_p^{(n)}(X)^p \sum_{i=1}^{n} \|y_i\|^p, \tag{3}$$

respectively

$$C_p^{(n)}(X)^p \mathbb{E}\|\sum_{i=1}^{n} r_i y_i\|^p \geq \sum_{i=1}^{n} \|y_i\|^p. \tag{4}$$

$T_p(X) := \lim_{n\to\infty} T_p^{(n)}(X)$ and $C_p(X) := \lim_{n\to\infty} C_p^{(n)}(X)$ are the type p and cotype p constants for X. The space X is said to be of type p; respectively, of cotype p, provided $T_p(X) < \infty$; respectively, $C_p(X) < \infty$.

For technical reasons, we also consider a weakened version of (L) which we term (WL); namely, that every quasi-nilpotent nuclear operator on X has trace zero. (The "technical reasons" are that in the unwritten paper [3] it is proved that (WL) implies (L) and we find it easier to verify that certain spaces satisfy (WL) rather than check that they satisfy (L)).

The weak Lidskii property (WL) implies the following property, which in turn implies that a Banach space X that satisfies (L) is an almost Hilbert space:

(HAP) X has the hereditary approximation property; that is, all of its subspaces have the AP.
(See [8] and the references therein).

Until now, the only spaces known to satisfy (L) are the weak Hilbert spaces (cf. [15, Chpt. 12], [16]; "weak Hilbert" is defined in the next section). Pisier built a beautiful theory of weak Hilbert spaces and there are some important weak Hilbert spaces, such as the 2 convexification of Tsirelson's space; [15, Chpt. 13]. However, the weak Hilbert spaces are somewhat elusive and there are very few known examples of them; in particular, Hilbert spaces are the only classical Banach spaces that are weak Hilbert spaces.

In this paper we exhibit a much larger class of spaces which satisfy the condition (WL). Nevertheless, our approach is a direct outgrowth of Pisier's approach to weak Hilbert spaces. We show that a Banach space that satisfies a weakened version of one of Pisier's equivalent conditions for being a weak Hilbert spaces must be a weak Lidskii space. It is relatively easy to show that many non weak Hilbert spaces, including some classical spaces other than Hilbert spaces, satisfy this weakened condition.

2. Γ-spaces

For $\varphi = (\varphi_1, \dots, \varphi_n) \in B^n_{X^*}$ and $x = (x_1, \dots, x_n) \in B^n_X$, let

$$G(\varphi, x) = \det[\langle \varphi_i, x_j \rangle]^n_{i,j=1}.$$

Define

$$G_n(X) = \sup\ \{|G(\varphi, x)| : \varphi \in B^n_{X^*}, x \in B^n_X\}. \tag{5}$$

$$\Gamma_n(X) = G_n(X)^{\frac{1}{n}}. \tag{6}$$

If $\dim E = n < \infty$ we denote

$$G(E) = G_n(E),\ \Gamma(E) = \Gamma_n(E).$$

Let us observe that

$$G(E) \geq 1 \ \text{ for } \ \text{every} \ \ E. \tag{7}$$

Indeed, let $\{x^*_i, x_i\}^n_{i=1}$ be an Auerbach system [11, Proposition 1.c.3] for E, i.e. $x^*_i \in B_{E^*}, x_i \in B_E$ and $x^*_i(x_j) = \delta_{ij}$. Then, clearly, $G(x^*_1, \dots, x^*_n; x_1, \dots, x_n) = 1$.

Define

$$\Gamma_{\sup}\ (X) = \sup \Gamma_n(X) \text{ and } \ \Gamma_{\inf}(X) = \liminf \Gamma_n(X) \tag{8}$$

Let us recall that a Banach space X is *a weak Hilbert (WH) space* if

$$\Gamma_{\sup}\ (X) < \infty.$$

This is one of several equivalences to a Banach space X being a WH space ([6, Theorem 15.1]).

We say that a Banach space X is a Γ-*space* provided

$$\Gamma_{\inf}(X) < \infty.$$

This is a substantial relaxation of the WH condition; nevertheless, as we show in this paper, the Γ-spaces still behave very much like WH spaces. In particular, they satisfy (L).

By $d_n(X)$ we denote the supremum over the n-dimensional subspaces E of X of the isomorphism constant from E to ℓ^n_2; that is, $d_n(X) = \sup\ \{d(E, l^n_2) : E \subset X, \dim E = n\}$. The isomorphism constant from E to F, $d(E, F)$, is the infimum of $\|T\| \cdot \|T^{-1}\|$ as T ranges over all isomorphisms from E onto F.

A few elementary facts concerning $G_n(X)$ and $\Gamma_n(X)$:

Lemma 2.1. *For any X, Y, n we have*

$$\Gamma_n(X) \le d(X,Y)\Gamma_n(Y) \tag{9}$$

$$G_1(X) \le G_2(X) \le \dots \tag{10}$$

$$G_n(X) \le d_n(X)G_{n-1}(X) \le n^{1/2}G_{n-1}(X). \tag{11}$$

Proof. (9) Let $T : X \to Y$ be an isomorphism. Given $\varphi = (\varphi_1, \dots, \varphi_n) \in B^n_{X^*}$ and $x = (x_1, \dots, x_n) \in B^n_X$, let us define $\psi_j = ((T^{-1})^*)\varphi_j, y_j = Tx_j$ for $j = 1, \dots, n$ and $\psi = (\psi_1, \dots, \psi_n), y = (y_1, \dots, y_n)$. We see that $G(\psi, y) = G(\varphi, x)$ and $\prod_{j=1}^n \|\psi_j\| \prod_{j=1}^n \|y_j\| \le (\|T\|\|T^{-1}\|)^n$, which implies (9).

(10). Let $\varphi = (\varphi_1, \dots, \varphi_n) \in B^n_{X^*}, x = (x_1, \dots, x_n) \in B^n_X$. Let $\varphi_{n+1} \in X^*$ be such that $\|\varphi_{n+1}\| = 1$ and span $\{x_1, \dots, x_n\} \subset \ker \varphi_{n+1}$, let $x_{n+1} \in B_X$ be such that $\varphi_{n+1}(x_{n+1}) = 1$.

Then $G(\varphi, x) = G(\varphi_1, \dots, \varphi_{n+1}, x_1, \dots, x_{n+1})$.

(11). Let us fix $x_1, \dots, x_n \in B_X$, $\varphi_1, \dots, \varphi_n \in B_{X^*}$. Let $E =$ span $\{x_1, \dots, x_n\}$, set $\psi_j = \varphi_{j|E}$, $\psi_j \in E^*$. Let $|\ \ |$ be a euclidean norm in E such that $d_n(X)^{-1}|x| \le \|x\| \le |x|$. Let us identify ψ_j with the $\psi_j \in E$ such that $\langle \psi_j, x_i \rangle = (\psi_j, x_i)$ for $i = 1, \dots, n$, where $(\ ,\ )$ is the scalar product corresponding to the norm $|\ \ |$.

For $y_1, \dots, y_k \in E$ let $V(y_1, \dots, y_k)$ denote the volume (induced by $(\ ,\ )$) of the parallelpiped $[0,1]y_1 + \dots + [0,1]y_k$.

We have $|G(\varphi, x)| = V(\psi_1, \dots, \psi_n)V(x_1, \dots, x_n)$. Since $|x_n| \le d_n(X)$ and $|\psi_n| \le 1$, we have $V(x_1, \dots, x_n) \le d_n(X)V(x_1, \dots, x_{n-1})$ and $V(\psi_1, \dots, \psi_n) \le V(\psi_1, \dots, \psi_{n-1})$. This gives the left inequality in (11). The right inequality in (11), that $d_n(X) \le n^{1/2}$, is a well known consequence of John's lemma about the maximal volume ellipsoid contained in the unit ball of a finite dimensional space [18, Theorem 6.30]). ■

Recall that a space X is *asymptotically Hilbertian* provided there are subspaces $Y_1, Y_2, \dots \subset X$ with $\dim X/Y_n < \infty$ and $\sup_n d_n(Y_n) < \infty$. Observe that we obtain the same definition if this is replaced by the formally weaker condition $\liminf_n d_n(Y_n) < \infty$ (this is so because $d_n(X)$ is always a non-decreasing sequence).

If, additionally, such Y_n can be chosen to be uniformly complemented in X, we say that X is *complementably asymptotically Hilbertian* (CAH).

We shall need the following fact

Proposition 2.1. *Let X be asymptotically Hilbertian. Then for every $\alpha > 0, d_n(X) = O(n^\alpha)$.*

Proof. Recall [13, Lemma 13.4] that the numbers $T_2^{(n)}$ and $C_2^{(n)}$ are submultiplicative; that is,

$$C_2^{(nm)}(Y) \le C_2^{(n)}(Y)C_2^{(m)}(Y), \quad T_2^{(n)}(Y) \le T_2^{(n)}(Y)T_2^{(m)}(Y). \tag{12}$$

In particular, for any natural number γ,

$$C_2^{(n^\gamma)}(Y) \le C_2^{(n)}(Y)^\gamma, \quad T_2^{(n^\gamma)}(Y) \le T_2^{(n)}(Y)^\gamma. \tag{13}$$

Consequently, using Kwapien's theorem [18, Theorem 13.15] in the first inequality below and the obvious inequality $\max\{C_2^n(Y), T_2^n(Y)\} \le d_n(Y)$ in the second, we get

$$d_{n^\gamma}(Y) \le (C_2^{(n)}(Y)T_2^{(n)}(Y))^\gamma \le d_n(Y)^{2\gamma}. \tag{14}$$

Let $\beta > 0$ be such that for every n there is a finite codimensional subspace $Y_n \subset X$ such that $d_n(Y_n) \le \beta$. Let m be such that $\beta^2 \le m^\alpha$. By (14), $d_{m^\gamma}(Y_m) \le \beta^{2\gamma} \le m^{\alpha\gamma}$. This clearly implies that $d_n(Y_m) = O(n^\alpha)$. Since $\dim X/Y_m < \infty$, also $d_n(X) = O(n^\alpha)$. ■

Remark 2.1. One cannot replace $O(n^\alpha)$ by e.g. $O(\log n)$: given $p_n \to 2$, consider $X = (\sum_{n=1}^{\infty} \oplus \ell_{p_n}^n)_2$. Then $d_n(X) \ge n^{|1/2-1/p_n|}$.

In section 3 we prove that Γ-spaces are complementably asymptotically Hilbertian.

In section 4 we prove that Γ-spaces satisfy condition (WL) (and hence, by [3] also satisfy (L)).

In section 5 we exhibit a large class of asymptotically Hilbertian spaces which are Γ-spaces, thus obtaining many spaces which satisfy condition (WL).

3. Γ-spaces are complementably asymptotically Hilbertian

Theorem 3.1. *Γ-spaces are complementably asymptotically Hilbertian.*

Let X be a Banach space and let E be a finite dimensional space. We will say that E is *infinitely reproducible in* X if for every $\varepsilon > 0$ there are $E_1, E_2, ... \subset X$ so that $E_1 \oplus E_2 \oplus \dots$ is a $(1+\varepsilon)$- Schauder decomposition (of $Z = \overline{E_1 \oplus E_2 \oplus \dots}$) and $d(E_i, E) \le (1+\varepsilon)$ for $i = 1, 2, ...$

Proposition 3.1. *Let Y be an ultraproduct of the form $Y = \prod Y_n/\mathcal{U}$ where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$ and Y_n is a decreasing sequence of finite codimensional subspaces of X with $\bigcup_n Y_n^\perp$ norm determining for X. Then every finite dimensional subspace of Y is infinitely reproducible in X.*

Proof. Let $E \subset Y$ be finite dimensional. There exist $F_n \subset Y_n$ such that $\lim_{n\in\mathcal{U}} d(F_n, E) = 1$. If F is a finite dimensional subspace of X that is $(1+\varepsilon)$-normed by $Y_n^\perp$, then the natural projection from $F+Y_n$ onto F has norm at most $(1+\varepsilon)$. Consequently, we can extract from the sequence $F_1, F_2, ...$ a subsequence $E_1, E_2, ...$ such that, on one hand, $E_1 \oplus E_2 \oplus \dots$ is a $(1+\varepsilon)$- Schauder decomposition and, on the other hand, $d(E_i, E) \le (1+\varepsilon)$ for $i = 1, 2, ...$ ∎

Proposition 3.2. *If E is infinitely reproducible in X, then*

$$\Gamma_{\inf}(X) \ge \Gamma(E). \tag{15}$$

Proof. Let $\dim E = n$. We claim that if n divides m, then

$$\Gamma_m(X) \ge \Gamma(E). \tag{16}$$

Indeed, let $\varepsilon > 0$, let $E_0, E_1, E_2, ... \subset X$ and $P_j : Z \to E_j$ (here $Z = \overline{E_0 \oplus E_1 \oplus E_2 \oplus \dots}$) be projections so that, for $j = 0, 1, 2, \dots,$

$$\|P_j\| \le 1+\varepsilon,\ \ P_jP_i = 0 \text{ if } i \ne j, \text{ and } d(E_j, E) \le (1+\varepsilon).$$

For $j = 0, 1, \dots,$ let $\varphi_1^j, \dots, \varphi_n^j \in B^n_{X^*}$ and $x_1^j, \dots, x_n^j \in B^n_{E_j}$ be such that $G(\varphi_1^j, \dots, \varphi_n^j; x_1^j, \dots, x_n^j) = G_n(E_j)$.

For $j = 0, 1, \dots, n-1$; $i = 1, \dots, n$, let $x_{jn+i} = x_i^j$ and let φ_{jn+i} be a Hahn-Banach extension to X of $P_j^*\varphi_i^j$.

Write $m = nk$, $k \in \mathbb{N}$. Since $\langle \varphi_{jn+i}, x_{ln+s}\rangle = \delta_{jl}\langle \varphi_i^j, x_s^j\rangle$, we have

$$\begin{gathered} G(\varphi_1, \dots, \varphi_m; x_1, \dots, x_m) = \prod_{j=1}^{k} G(\varphi_1^j, \dots, \varphi_n^j; x_1^j, \dots, x_n^j) \ge \\ \prod_{j=1}^{k} G(E_j) \ge \prod_{j=1}^{k}[d(E_j, E)^{-1}G(E)] \ge (1+\varepsilon)^{-k}G(E)^k. \end{gathered} \tag{17}$$

Hence $G_m(X) \ge (1+\varepsilon)^{-k}G(E)^k$, thus $G_m(X) \ge G(E)^k$ and (16) follows. The proposition follows from (16), because if $t = jn+i$, then $G_t(X) \ge G_{jn}(X) \ge \Gamma(E)^{jn}$, thus $\Gamma_t(X) \ge \Gamma(E)^{\frac{jn}{t}}$ and this goes to $\Gamma(E)$ when $t \to \infty$.

∎

Recall that, given a property (P), a space X has property *asymptotically-P*, denoted (as. P), if there is a sequence $Y_n \subset X$ of subspaces of finite codimension and a free ultrafilter on $\mathbb{N}$ such that the ultraproduct $Y = \prod Y_n/\mathcal{U}$ has the property (P).

From propositions 3.1 and 3.2 we obtain the following

Corollary 3.1. *Γ-spaces are asymptotically WH.*

It is, however, well known that if an ultraproduct is an as.WH space, then it is (isomorphic to) an assymptotically Hilbert space [16, chap. 14]. This proves the "asymptotically Hilbertian" part of Theorem 3.1. To get the "complementably" part of the theorem, we need the following proposition, which is an adaptation of a result due to Maurey and Pisier. For the convenience of the reader we reproduce a proof of it from [12].

Proposition 3.3. *Let X be a reflexive Banach space. For every subspace $Y \subset X$ of co-dimension n and for every $\varepsilon > 0$ with $m = (1+\varepsilon)n \in \mathbb{N}$ there exists a projection $Q : X \to X$ whose co-rank is at most m such that $QX \subset Y$ and $\|Q\| \leq [G_{m+1}^{1/m}(X)]^{\frac{1+\varepsilon}{\varepsilon}}$.*

Proof. For $E \subset X, F \subset X^*$ let us denote

$$G(E, F) = \sup \{|G(\varphi, x)| : \varphi \in B_F^n, x \in B_E^n\}.$$

For $k = n, n+1, \dots, m+1$ we shall define by induction k-dimenensional subspaces E_k of X and F_k of X^*,

$$E_n \subset E_{n+1} \subset \cdots \subset E_{m+1},\ F_n \subset F_{n+1} \subset \cdots \subset F_{m+1}$$

as well as a sequence of projections $P_n : X \to X$.

Let $F_n = Y^\perp$ and let $E_n \subset X$ be such that $\dim E_n = n$ and $G(E_n, F_n) = 1$ (it exists, by the reflexivity and by (7)).

Let $m \geq k \geq n$, and assume that we have defined $E_k \subset X, F_k \subset X^*$. Let $P_k : X \to X$ be the projection onto E_k with $\ker P_k = F_k^\perp$. Let $x_i^* \in B_{X^*}, x_i \in B_X, i = 1, \dots, k$ be such that $G(E_k, F_k) = G(x_1^*, \dots, x_k^*; x_1, \dots, x_k)$.

We define $x_{k+1} \in B_X$ and $x_{k+1}^* \in B_{X^*}$ so that

$$\langle x_{k+1}^*, x_{k+1} - P_k(x_{k+1})\rangle = \|I_X - P_k\|.$$

Since $P_k(x_{k+1}) \in \text{span}\ \{x_1, \dots, x_k\}$, the determinant of the matrix $[\langle x_i^*, x_j\rangle]_{i,j=1}^{k+1}$ does not change if its last column (corresponding to x_{k+1}) is replaced by the vector $(0, \dots, 0, \langle x_{k+1}^*, x_{k+1} - P_k(x_{k+1})\rangle)$. We set $F_{k+1} = \text{span}\ \{x_1^*, \dots, x_{k+1}^*\}, E_{k+1} = \text{span}\ \{x_1, \dots, x_{k+1}\}$. Consequently, we have

$$\begin{aligned} G(E_{k+1}, F_{k+1}) &\geq \det[\langle x_i^*, x_j\rangle]_{i,j=1}^{k+1} = \\ &\langle x_{k+1}^*, x_{k+1} - P_k(x_{k+1})\rangle G(E_k, F_k) = \|I_X - P_k\| G(E_k, F_k). \end{aligned} \tag{18}$$

Since $G(E_n, F_n) = 1$, we have for $m \geq k \geq n$

$$G(E_{k+1}, F_{k+1}) \geq \prod_{j=n}^{k} \|I_X - P_j\|.$$

For $k = m$, the right-hand side has εn elements, thus at least one of them is less than $G_{m+1}(X)^{\frac{1}{\varepsilon n}}$, say $\|I_X - P_j\| \leq G_{m+1}(X)^{\frac{1}{\varepsilon n}} = [G_{m+1}^{1/m}(X)]^{\frac{1+\varepsilon}{\varepsilon}}$. We set $Q = I_X - P_j$. ■

Remark 3.2. It is clear that very little happens if we drop the assumption of reflexivity: the assertion is true with $\|Q\| < \gamma$ where γ is any number larger than $[G_{m+1}^{1/m}(X)]^{\frac{1+\varepsilon}{\varepsilon}}$.

4. Γ-spaces are Lidskii spaces

We shall apply Fredholm determinant theory, as presented in [16, chap. 15]. Let us recall some basic notions and facts.

For $n = 1, 2, \dots$ and $x_1, x_2, \dots \in X$, $x_1^*, x_2^*, \dots \in X^*$ we set

$$\alpha_n(x_1^* \otimes x_1, \dots, x_n^* \otimes x_n) = \frac{1}{n!} \det([\langle x_j^*, x_i \rangle])$$

and for $T_1, \dots, T_n \in F(X)$ we define $\alpha_n(T_1, \dots, T_n)$ by the n-linear extension.

Lemma 4.1. [16, Prop. 15.3.i]

$$\alpha_n(T_1, \dots, T_n) \leq \frac{G_n(X)}{n!} \|T_1\|_\wedge \dots \|T_n\|_\wedge. \tag{19}$$

Now, by continuity, we can extend α_n to any $T_1, \dots, T_n \in N(X)$. For $T \in N(X)$ denote $\alpha_n(T) = \alpha_n(T, \dots, T)$. Let us observe that if $T = \sum_{i=1}^{\infty} x_i^* \otimes x_i$ with $\sum \|x_i^*\| \|x_i\| < \infty$, then $\alpha_1(T) = \operatorname{tr} T (= \sum_{i=1}^{\infty} x_i^*(x_i))$. This is so, because α_1 is a continuous linear functional on $N(X)$ and $\alpha_1(x^* \otimes x) = x^*(x)$.

Since $G_n(X) \leq d_n(X)^n \leq n^{n/2}$, we get for every $T \in N(X)$,

$$\alpha_n(T) \leq \frac{n^{n/2}}{n!} \|T\|_\wedge^n.$$

Thus the series $\sum \alpha_n(T)$ converges absolutely for every $T \in N(X)$ and we set

$$\det(I + T) = \sum_{n=0}^{\infty} \alpha_n(T) \quad \text{with } \alpha_0(T) = 1$$

and for $z \in \mathbb{C}$ we define

$$D_T(z) = \det(I + zT) = \sum_{n=0}^{\infty} \alpha_n(T) z^n.$$

$D_T(z)$ is obviously an entire function for every $T \in N(X)$. It is well known that the zeros of D_T are precisely the inverses of the non-zero eigenvalues of T.

Let us observe that the order of the entire function D_T is at most 2. Indeed, by a basic formula (cf. [1, Theorem 2.2.2.]), if $f(z) = \sum \alpha_n z^n$, then the order of f, $\varrho(f)$ is equal to $[\limsup \frac{n \log n}{\log(\frac{1}{|\alpha_n|})}]$. By (19) and by Stirling's formula,

$$\begin{aligned} \log(\frac{1}{|\alpha_n|}) \geq \log(\frac{n!}{d_n(X)^n \|T\|_\wedge^n}) \geq \\ n(\log n - 1 - \log \|T\|_\wedge - \log d_n(X)) \geq \frac{1}{2} n \log n - o(n \log n), \end{aligned} \tag{20}$$

thus $\varrho(D_T) \leq 2$.

Proposition 4.1. *Assume that $d_n(X) = o(n^\gamma)$ with $\gamma < \frac{1}{2}$. If $T \in N(X)$ and T is quasi-nilpotent, then $D_T(z) = \exp(az)$ for some $a \in \mathbb{C}$.*

Proof. By (20), $\log(\frac{1}{|a_n|}) \geq (1-\gamma) n \log n - o(n \log n)$, thus $\varrho(D_T) \leq \frac{1}{1-\gamma} < 2$, hence $\varrho(D_T) \leq 1$.

T being quasi-nilpotent, D_T does not have any zeros, thus by the Hadamard factorization theorem [1, Theorem 2.7.1.], $D_T(z)$ must have the form $\exp(az+b)$ with some $a, b \in \mathbb{C}$.

Since $D_T(0) = 1$, we have $\exp b = 1$. ■

Lemma 4.2. [16, Lemma 15.4] *For every $T \in N(X)$ and every $\varepsilon > 0$ there exists $C = C_\varepsilon(T)$ such that for every n,*

$$|\alpha_n(T)| \leq C \frac{G_n(X)}{n!} \varepsilon^n. \tag{21}$$

Proof. Let $U \in F(X)$ be such that $\|T - U\|_\wedge \leq \frac{\varepsilon}{2}$. Set $V = T - U$. We have

$$\alpha_n(T) = \sum_{j=0}^{n} \binom{n}{j} \alpha_n(U, \ldots, U, V, \ldots, V)$$

(U j times). Let $k = \mathrm{rk}\, U$, thus for $n \geq k$ we have

$$\alpha_n(T) = \sum_{j=0}^{k} \binom{n}{j} \alpha_n(U, \ldots, U, V, \ldots, V).$$

By (19),

$$\begin{aligned} |\alpha_n(U, \ldots, U, V, \ldots, V)| &\leq \frac{G_n(X)}{n!} \|U\|_\wedge^j \|V\|_\wedge^{n-j} \\ &\leq \frac{G_n(X)}{n!} \varepsilon^n 2^{-n} 2^j \varepsilon^{-j} \|U\|_\wedge^j, \end{aligned} \tag{22}$$

thus (21) holds with $C = \max_n 2^{-n} \sum_{j=0}^{k} \binom{n}{j} 2^j \varepsilon^{-j} \|U\|_\wedge^j$. ■

As a corollary we obtain

Theorem 4.1. *If X is a Γ-space, then X is a weak Lidskii space; i.e., X satisfies (WL).*

Proof. Let $K < \infty$ be such that $G_n(X) \le K^n$ for infinitely many n. We know that X is asymptotically Hilbertian, thus $d_n(X) = o(n^\gamma)$ for every $\gamma > 0$, by Proposition 2.1. Let $T \in N(X)$ be quasi-nilpotent. By Proposition 4.1, $D_T(z) = \exp(az)$ for some $a \in \mathbb{C}$, hence $|\alpha_n(T)| = \frac{|a|^n}{n!}$. Fix an $\varepsilon > 0$. By Lemma 3, $|a| \le C_\varepsilon^{\frac{1}{n}} K\varepsilon$ for infinitely many n, thus $a = 0$. This shows that formula (L) holds for all quasi-nilpotent operators in $N(X)$. By [3], this implies that X is a weak Lidskii space. ■

Combining Theorem 4.1 with the result from [3] that (WL) implies (L) we get

Corollary 4.1. *If X is a Γ-space, then X is a Lidskii space.*

Since [3] has yet to be written, we sketch a proof of

Theorem 4.2. (Figiel-Johnson) *If X satisfies* (WL)*, then X satisfies* (L)*.*

Proof. The main tool is Ringrose's [17] structure theory for compact operators. Let T be a compact operator on a complex Banach space X and let $\mathcal{N}$ be a maximal nest of closed subspaces of X that are invariant for T. Given $N \in \mathcal{N}$, let N^- be the closed linear span of all M in $\mathcal{N}$ that are properly contained in N. Ringrose observes that either $N^- = N$ or N^- has codimension one in N. In the latter case, there is an eigenvalue λ_N of T so that for every $x \in N \sim N^-$ we have $Tx = \lambda_n x + y_x$ with $y_x \in N^-$. The collection $\mathcal{N}' := \{\lambda_N : \dim N/N^- = 1\}$ exhausts the eigenvalues of T repeated according to multiplicity, and so the collection $\mathcal{N}'$ is countable.

Suppose now that $\sum_{N\in\mathcal{N}'} |\lambda_N| < \infty$. For $N \in \mathcal{N}$ pick $x_N \in N$ of norm one so that the distance of x_N to N^- is close to one. choose a functional $x_N^* \in (N^-)^\perp$ with norm close to one so that $x_N^*(x_N) = 1$. Then the linear operator $S := \sum_{N\in\mathcal{N}'} \lambda_n x_N^* \otimes x_N$ is nuclear and every $N \in \mathcal{N}$ is an invariant subspace for S. Consequently, $\mathcal{N}$ is a (necessarily maximal) nest of invariant subspaces for the compact operator $T - S$. By construction, for every $N \in \mathcal{N}'$ we have that $(T - S)N \subset N^-$, which is to say that $T - S$ is quasi-nilpotent, and of course nuclear if T is nuclear. ■

5. Examples of Γ-spaces

We shall say that X is *asymptotically Hilbertian of polynomial growth* if there is a constant λ such that there are subspaces $Y_1, Y_2, \dots \subset X$ with $\dim X/Y_n = O(n^\lambda)$ and $\liminf d_n(Y_n) < \infty$.

Proposition 5.1. *In the above definition, "there is λ" implies "for every $\lambda > 0$".*

Proof. Let $n_1 < n_2 < \dots$ be a sequence such that for every j, $d_{n_j}(Y_{n_j}) \le d < \infty$ and $\dim X/Y_{n_j} \le Cn_j^\lambda$. Let $m_j = n_j^\gamma$ and let $Z_{m_j} = Y_{n_j}$ for $j = 1, 2, \dots$. By (14), we have $d_{m_j}(Z_{m_j}) \le d^{2\gamma}$ whereas $\dim X/Z_{m_j} \le Cm_j^{\frac{\lambda}{\gamma}}$. Taking γ sufficiently large, we get $\frac{\lambda}{\gamma}$ as small as we wish.

■

Theorem 5.1. *If X is complementably asymptotically Hilbertian of polynomial growth, then X is a Γ-space.*

Theorem 5.1 follows from

Lemma 5.1. *Let $P : X \to X$ be a rank k projection and set $Y = \ker P$. Denote $K = \max(\|P\|, \|I_X - P\|)$. Then*

$$G_n(X) \le K^{2n} n^{2k} d_n(Y)^n; \textit{ i.e. } \ \Gamma_n(X) \le K^2 n^{\frac{2k}{n}} d_n(Y).$$

Proof. For $\varphi \in X^*$, $x \in X$ denote

$$\varphi^1 = P^*\varphi, \varphi^0 = (I_X - P^*)\varphi, x^1 = Px, x^0 = (I_X - P)x.$$

For $\varepsilon \in \{0,1\}^n$ let $|\varepsilon| = \sum_{j=1}^n \varepsilon_j$.

For $\varepsilon, \eta \in \{0,1\}^n, \varphi \in X^{*n}, x \in X^n$ denote $G_{\varepsilon,\eta}(\varphi, x) = \det[\langle \varphi_i^{\eta(i)}, x_j^{\varepsilon(j)} \rangle]$. By the $2n$-linearity of $G(\varphi, x)$, we have

$$G(\varphi, x) = \sum_{\varepsilon,\eta \in \{0,1\}^n} G_{\varepsilon,\eta}(\varphi, x).$$

Clearly $\langle \varphi_i^1, x_j^0 \rangle = \langle \varphi_i^0, x_j^1 \rangle = 0$ for all i, j, therefore $G_{\varepsilon,\eta}(\varphi, x) = 0$ unless $|\varepsilon| = |\eta|$. For $A, B \subset \{1, \dots, n\}$ such that $|A| = |B|$, let $G^1_{A,B}(\varphi, x)$ be the minor of the matrix $[\langle \varphi_i^1, x_j^1 \rangle]_{i,j=1}^n$, corresponding to the rows in A and to the columns in B and let $G^0_{A,B}(\varphi, x)$ be the minor of the matrix $[\langle \varphi_i^0, x_j^0 \rangle]_{i,j=1}^n$, corresponding to the rows in A^c and to the columns in B^c. We see that

$$G_{\varepsilon,\eta}(\varphi, x) = \sigma(A, B) G^1_{A,B}(\varphi, x) G^0_{A,B}(\varphi, x),$$

where $\sigma(A,B)=\pm 1$ and $A=\{i:\varepsilon(i)=1\}$, $B=\{j:\varphi(j)=1\}$. Since $\dim PX=k$, we have $G^1_{A,B}=0$ for $|A|=|B|>k$, thus

$$G(\varphi,x)=\sum_{A,B\subset\{1,\dots,n\},|A|=|B|\le k}\sigma(A,B)G^1_{A,B}(\varphi,x)G^0_{A,B}(\varphi,x).$$

Let us observe that for $|A|=|B|=j$ we have

$$|G^1_{A,B}(\varphi,x)|\le\|P\|^{2j}G_j(PX), |G^0_{A,B}(\varphi,x)|\le\|I_X-P\|^{2(n-j)}G_{n-j}(Y),$$

therefore

$$|G(\varphi,x)|\le K^{2n}\sum_{j=0}^{k}\binom{n}{j}^2 G_j(PX)G_{n-j}(Y).$$

By Lemma 2.1 we have

$$G_j(PX)\le j!^{1/2}\ \text{ and }\ G_{n-j}(Y)\le G_n(Y),$$

thus

$$|G(\varphi,x)|\le K^{2n}n^{2k}G_n(Y)\le K^{2n}n^{2k}d_n(Y)^n,$$

since $G_n(Y)\le d_n(Y)^n$. ■

Proof of Theorem 5.1. By the definition, there are $\beta,K,C<\infty$ and Y_n for $n=1,2,3,\dots$ so that $q_n=\dim X/Y_n<\infty$ and (i),(ii),(iii) below are satisfied:

(i) $d_n(Y_n)\le\beta$,

(ii) $\liminf q_n\log n/n<\infty$,

(iii) there are projections $P_n:X\to Y_n$ with $\|P_n\|,\|I_X-P_n\|\le K$

((ii) follows from Proposition 5.1). By Lemma 5.1, $\Gamma_{q_n}(X)\le K^2\beta e^{\frac{3}{2}C}$. ■

The primary example are spaces of the form $X=(\sum_{n=1}^{\infty}\oplus\ell^{k_n}_{p_n})_2$.

Denote $\delta_n=|p_n-2|$. We have for $n\le m, d_n(\ell^m_p)=n^{|\frac12-\frac1p|}$, thus $\frac16|p-2|\le d_n(\ell^m_p)\le\frac12|p-2|$ for $1\le p\le 3, n\le m$. Therefore

(*) X is not isomorphic to a Hilbert space iff sup $k_m^{\delta_m}=\infty$.

Let us make two ad hoc assumptions:

$$\delta_n\searrow 0,\ k_n\ge 2k_{n-1}\text{ for every }n.$$

Set $Z_m=\{0\}\oplus\cdots\oplus\{0\}\oplus(\sum_{n=m+1}^{\infty}\oplus\ell^{k_n}_{p_n})_2$. Then

$$d_n(Z_m)\le d_n(\ell_{p_{m+1}})=n^{\delta_{m+1}},\ \text{codim }(Z_m)=k_1+\cdots+k_m\le 2k_m.$$

The next two lemmas follow now from elementary computations:

Lemma 5.2. *X is asymptotically Hilbertian of polynomial growth provided the sequence $\{k_m^{\delta_{m+1}}\}$ is bounded.*

Together with (*) this gives

Lemma 5.3. *If $\frac{\log k_{m+1}}{\log k_m} \to \infty$ and $\delta_{m+1} = \frac{C}{\log k_m}$, then X is complementably asymptotically Hilbertian of polynomial growth and not isomorphic to a Hilbert space.*

An immediate consequence of the preceding lemma and the constructions and results in [2] we get

Theorem 5.2. *The direct sum of two Lidskii spaces need not be a Lidskii space.*

In fact, if $p_n \to 2$ and $k_n \to \infty$ and $X = (\sum_{n=1}^{\infty} \oplus \ell_{p_n}^{k_n})_2$, we can get a partition $N = N_1 \cup N_2$ so that both $X_1 := (\sum_{n \in X_1} \oplus \ell_{p_n}^{k_n})_2$ and $X_2 := (\sum_{n \in X_2} \oplus \ell_{p_n}^{k_n})_2$ are complementably asymptotically Hilbertian of polynomial growth, while $X = X_1 \oplus X_2$ can even fail to be HAPpy if the rates $p_n \to 2$ and $k_n \to \infty$ are chosen appropriately–see [2].

6. Open questions

Question 1. Suppose $G_n(X)$ is bounded or $G_n(X)^{\frac{1}{n}} \to 1$. Does it follow that X is (isomorphic to) a Hilbert space?

Question 2. If X is isomorphic to a Hilbert space, does it follow that $G_n(X)$ is bounded or that $G_n(X)^{\frac{1}{n}} \to 1$?

Question 3. Is (HAP) equivalent to (L)?

Question 4. Is every asymptotically Hilbertian space CAH?

Question 5. Suppose that $d_n(X)$ goes to infinity sufficiently slowly. Must X be a Lidskii space?

References

[1] R. Boas, Entire Functions, New York : Academic Press, 1954

[2] P. G. Casazza, C. L. Garcia, and W. B. Johnson, An example of an asymptotically Hilbertian space which fails the approximation property, Proc. AMS 129 (2001), 3017-3023.

[3] T. Figiel and W. B. Johnson, in preparation.

[4] A. Grothendieck, Produits Tensoriels Topologiques et Espaces Nucleaires, Memoirs AMS 16 (1955).

[5] A. Grothendieck, La théorie de Fredholm, Bull. Soc. Math. France 84 (1956), 319–384.

[6] W. B. Johnson, Banach spaces all of whose subspaces have the approximation property, (Proc. Sem., Ges. Math. Datenverarb., Bonn, 1979), North-Holland, Amsterdam-New York, (1980), 15–26.

[7] W. B. Johnson, H. König, B. Maurey, J. R. Retherford, Eigenvalues of p-summing and l_p-type operators in Banach spaces, J. Funct. Anal. 32 (1979), 353–380.

[8] W. B. Johnson and A. Szankowski, Hereditary approximation property, Annals of Math. 176 (2012), 1987–2001.

[9] V. Lidskii, Non-self adjoint operators with a trace, Dokl.Akad.Nauk SSSR 125 (1959), 485–487.

[10] J. Lindenstrauss and L. Tzafriri, On the complemented subspaces problem, Israel J. Math. 9 (1971), 263–269.

[11] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces I, Sequence spaces, Springer, Berlin 1977.

[12] V. Mascioni, On Banach spaces isomorphic to their duals, Houston J. Math. 19 (1993), 27-38

[13] V. Milman and G. Schechtman, Asymptotic theory of finite dimensional normed spaces. Springer-Verlag, Lecture Notes in Mathematics 1200, (1986).

[14] G. Pisier, Martingales with values in uniformly convex spaces, Israel J. Math. 20 (1975), 326–350.

[15] G. Pisier, Weak Hilbert spaces, Proc. London Math. Soc. (3) 56 (1988), no. 3, 547–579.

[16] G. Pisier, The volume of convex bodies and Banach space geometry, Cambridge Tracts in Math. 94 (1989).

[17] J. R. Ringrose, Super-diagonal forms for compact linear operators , Proc. London Math. Soc. (3), 12 (1962), 367–384.

[18] N. Tomczak-Jaegermann, Banach-Mazur distances and finite-dimensional operator ideals , Longman Scientific and Technical, 1989.

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