

Differential geometry Math 622 Homework 6 (2 questions!)

Due 2/27/8 in class

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Let $X \in \Gamma(TM)$ be a vector field. While we already know how to differentiate functions in the direction of X , the *Lie derivative*, denoted by \mathcal{L}_X , enables us to differentiate vector fields, differential forms, or sections of any of the tensor product bundles $(T^*M)^{\otimes k} \otimes (TM)^{\otimes l}$.

The Flowbox Theorem implies that on a sufficiently small neighborhood U of any point in M , there exists a family of local diffeomorphisms $\phi_t : U \rightarrow M$ such that $\phi_0 = Id$ and for any $x \in U$, $t \mapsto \phi_t(x)$ is an integral curve of X . For $\alpha \in \Omega^k(M)$, we define $\mathcal{L}_X \alpha \in \Omega^k(M)$ by

$$\mathcal{L}_X \alpha = \lim_{t \rightarrow 0} \frac{1}{t} [\phi_t^* \alpha - \alpha].$$

For $Y \in \Gamma(TM)$, we define $\mathcal{L}_X Y \in \Gamma(TM)$ by

$$\mathcal{L}_X Y = \lim_{t \rightarrow 0} \frac{1}{t} [(\phi_{-t})_* Y - Y].$$

We extend the Lie derivative to $\Gamma((T^*M)^{\otimes k} \otimes (TM)^{\otimes l})$ by the Leibniz rule. For example, for $\alpha, \beta \in \Omega^1(M)$,

$$\mathcal{L}_X(\alpha \otimes \beta) = (\mathcal{L}_X \alpha) \otimes \beta + \alpha \otimes \mathcal{L}_X \beta.$$

Similarly, $\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X \beta$.

1. Show that $\mathcal{L}_X Y = [X, Y]$.
2. Show that $\mathcal{L}_X \alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha)$. (Hint: Prove for 1-forms and extend using wedge products.)