Subject §3.6 Row and Column Spaces.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$.

consider its row vectors and column vectors.

Def: The subspace of $\mathbb{R}^{1\times n}$ spanned by the row vectors of A is called the row space of A. denoted by $RS(A) \subseteq \mathbb{R}^n$.

The subspace of $\mathbb{R}^{m\times 1}$ spanned by the column vectors of A is called the column space of A, denoted by $CS(A) \subseteq \mathbb{R}^m$.

Ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

$RS(A) = \text{span}\{(1,0,0), (0,1,0)\} = \{(\alpha, \beta, 0)\}$.

$CS(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \left\{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right\}$.

consider the reduced Echelon form.

THM: Any two row equivalent matrices have the same row space and linear relation among the columns.

* Columns with leading 1's are LI. A column without the leading 1 is a linear combination of columns with leading 1 to its left.

Def: Given $A_{m \times n}$, the rank r of A is the dimension of the row space, $r = \dim(RS(A))$.

* to determine the rank, $A \to \cdots \to U$ reduced row Echelon form.

r = the # of rows with leading 1 = # of leading 1.

Ex. $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & -5 \\ 0 & -2 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \end{matrix}$

$ \uparrow \ \uparrow$

$RS(A) = \text{span}\{(1,-2,3), (2,-5,1)\}$. $CS(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ -4 \end{bmatrix}\right\}$.

Note $CS(A) \neq \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\}$.

consider $Ax = b \Leftrightarrow$ linear combination of columns of $A = b$.

So $Ax = b$ is consistent if and only if $b \in CS(A)$.

As for $Ax = 0$, it has only zero solution if and only if the columns of A are LI.

THM: Consider $A_{m \times n}$.

1) $Ax = b$ is consistent for each $b \in \mathbb{R}^m$ if and only if the column vectors of A span $\mathbb{R}^m$.

2) $Ax = 0$ has only the zero solution if and only if the column vectors of A are LI.

Corollary: An $n\times n$ matrix A is nonsingular if and only if the column vectors of A form a basis for $\mathbb{R}^n$.

*THM. Let $A_{m\times n}$, $r = \text{rank}(A)$, k = nullity of $A = \dim(N(A))$. then $\qquad n = r + k$

Proof: Consider $Ax = 0$. $A \xrightarrow{\text{reduced Echelon form}} U = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\quad}^{r} & \overbrace{\qquad\quad}^{n-r} \end{array} \\ \left[\begin{array}{ccc:ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hdashline 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right] \begin{array}{l} \}r \\ \\ \\ \}m-r \end{array} \end{array}$

n = # of unknowns, r = # of leading 1's

$n-r$ = # of ~~rows~~ columns without leading 1 = Freedom

$= \dim(N(A)) = k \Rightarrow n = r + k$

* Columns with leading 1's are LI.
* A column without a leading 1 is a linear combination of columns with leading 1 to its left.

* r = # of leading 1's = rank(A) = $\dim(RS(A))$
$=$ # of columns with leading 1 $= \dim(CS(A))$

Ex. Let $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & 3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$

1) Find a basis for $RS(A)$. Determine $r = \dim(RS(A))$
2) Find a basis for $CS(A)$.
3) Find a basis for $N(A)$. Determine $k = \dim(N(A))$.
4) verify $n = r + k$.

1) $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow RS(A) = \text{span}\left\{ \begin{matrix} (1,2,-1,1) \\ (2,4,-3,0) \end{matrix} \right\} = \text{span}\left\{ \begin{matrix} (1,2,0,3) \\ (0,0,1,2) \end{matrix} \right\}. \quad r = 2$

2)

$\Rightarrow CS(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\} \neq \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$

3). set $x_4 = s$, $x_3 = -2s$, $x_2 = t$, $x_1 = -2t - 3s$.

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t-3s \\ t \\ -2s \\ s \end{bmatrix} = s\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$

$N(A) = \text{span}\left\{ \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}. \quad k = 2 = \dim(N(A))$

4). $n = 4 = r + k = 2 + 2.$

Let $A_{m\times n}$, $N(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}$

Given $b \in \mathbb{R}^m$, consider $Ax = b$.

If there is $x_p \in \mathbb{R}^n$ such that $Ax_p = b$.

x_p is called a particular solution to $Ax = b$, then the solution space to $Ax = b$ is

$$x_p + N(A) = \{ x_p + x_0 : x_0 \in N(A) \}$$

<u>Proof</u>: Let x_1 be any solution to $Ax = b$, we show that $x_1 = x_p + x_0$ for some $x_0 \in N(A)$. Denote $x_0 = x_1 - x_p$

we have $Ax_0 = Ax_1 - Ax_p = b - b = 0 \Rightarrow x_0 \in N(A)$. #.

Ex: Find the solution space to

$$\begin{aligned} x_1-4x_2+7x_3&=-4\\ x_1-x_2+x_3&=2\\ x_1-2x_2+3x_3&=0 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -4 & 7 & -4\\ 1 & -1 & 1 & 2\\ 1 & -2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -4 & 7 & -4\\ 0 & 3 & -6 & 6\\ 0 & 2 & -4 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -4 & 7 & -4\\ 0 & 1 & -2 & 2\\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4\\ 0 & 1 & -2 & 2\\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3=t,\ x_2=2+2t,\ x_1=4+t.$

$$\begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix} = \begin{bmatrix} 4+t\\ 2+2t\\ t \end{bmatrix} = \begin{bmatrix} 4\\ 2\\ 0 \end{bmatrix} + t\begin{bmatrix} 1\\ 2\\ 1 \end{bmatrix}, \quad x_p=\begin{bmatrix} 4\\ 2\\ 0 \end{bmatrix}. \quad N(A)=\operatorname{span}\left\{\begin{bmatrix} 1\\ 2\\ 1 \end{bmatrix}\right\}.$$

Ex: $[A|b] \to \cdots \to$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & 3 & -2\\ -2 & 4 & 1 & 2 & -4 & 9\\ 3 & -6 & 1 & 7 & 11 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & 3 & -2\\ 0 & 0 & 1 & 4 & 2 & 5\\ 0 & 0 & 1 & 4 & 2 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & 3 & -2\\ 0 & 0 & 1 & 4 & 2 & 5\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \begin{bmatrix} -2\\ 5\\ 0\\ 0 \end{bmatrix} = -2\begin{bmatrix} 1\\ 0\\ 0\\ 0 \end{bmatrix} + 5\begin{bmatrix} 0\\ 1\\ 0\\ 0 \end{bmatrix}.$$

If $A=[a_1\, a_2\, a_3\, a_4\, a_5]$, $a_1=\begin{bmatrix} 1\\ 1\\ 3\\ 4 \end{bmatrix}$, $a_3=\begin{bmatrix} 2\\ -1\\ 1\\ 3 \end{bmatrix}$.

then $b=-2a_1+5a_3=-2\begin{bmatrix} 1\\ 1\\ 3\\ 4 \end{bmatrix}+5\begin{bmatrix} 2\\ -1\\ 1\\ 3 \end{bmatrix}=\begin{bmatrix} 8\\ -7\\ -1\\ 7 \end{bmatrix}$.

Recover a_2, a_4, a_5.

a_2=-2a_1, a_4=a_1+4a_3, a_5=3a_1+2a_3, b=-2a_1+5a_3