Subject §4.2 Matrix Representation

Given $\dim(V) = n$, $\dim(W) = m$, $L : V \to W$ linear.

Let $E = \{v_1, \cdots, v_n\}$ be a basis for V and

$F = \{w_1, \cdots, w_m\}$ be a basis for W.

Let $v = x_1 v_1 + \cdots + x_n v_n$ be any given vector in V.

$\Leftrightarrow [v]_E = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$,

$L(v) = L(x_1 v_1 + \cdots + x_n v_n) = x_1 L(v_1) + \cdots + x_n L(v_n) \in W$

$= y_1 w_1 + \cdots + y_m w_m \Leftrightarrow [L(v)]_F = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^m$.

Try to show that there is a $A_{m \times n}$ such that

$$[L(v)]_F = A\,[v]_E$$

where $A = \left[\,[L(v_1)]_F \cdots [L(v_n)]_F\,\right]$ is called the matrix representation of L w.r.t. the basis E for V the basis F for W.

Since $L(v_j) \in W$, denote

$L(v_j) = a_{1j} w_1 + a_{2j} w_2 + \cdots + a_{mj} w_m = \sum_{i=1}^{m} a_{ij} w_i$,

$\Leftrightarrow [L(v_j)]_F = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m$. $j = 1, 2, \cdots, n$.

$$L(v)=\sum_{j=1}^{n} x_j L(v_j)=\sum_{j=1}^{n} x_j \sum_{i=1}^{m} a_{ij} w_i=\sum_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij} x_j\Big) w_i$$

$$\Rightarrow [L(v)]_F=\begin{bmatrix}\sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j\end{bmatrix}=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ & \cdots & \\ a_{m1} & \cdots & a_{mn}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n\end{bmatrix}=A[v]_E$$

So $\underbrace{A=\Big[[L(v_1)]_F \cdots [L(v_n)]_F\Big]}$ $\Rightarrow [L(v)]_F=A[v]_E \quad v\in V.$

Diagram:

$$\begin{tikzcd} v\in V \arrow[r, "L"] \arrow[d, "E"'] & W \arrow[d, "F"] \\ {[v]_E\in \mathbb{R}^n} \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

$w=L(v)\in W$

$[L(v)]_F=A[v]_E\in\mathbb{R}^m$

Ex. Given $L:\mathbb{R}^3\to\mathbb{R}^2$ by $L(x)=x_1 b_1+(x_2+x_3)b_2$ where $b_1=\begin{bmatrix}1\\1\end{bmatrix}$, $b_2=\begin{bmatrix}-2\\1\end{bmatrix}$, $B=\{b_1,b_2\}$ is a basis for $\mathbb{R}^2$.
Find the representation matrix A of L w.r.t. the basis $\{e_1,e_2,e_3\}$ in $\mathbb{R}^3$ and $B=\{b_1,b_2\}$ in $\mathbb{R}^2$.

We have $A=\Big[[L(e_1)]_B\,[L(e_2)]_B\,[L(e_3)]_B\Big]$.

$L(e_1)=1 b_1+(0+0)b_2 \Rightarrow [L(e_1)]_B=\begin{bmatrix}1\\0\end{bmatrix}$

$L(e_2)=0\cdot b_1+(1+0)b_2 \Rightarrow [L(e_2)]_B=\begin{bmatrix}0\\1\end{bmatrix}$

$L(e_3)=0\cdot b_1+(0+1)b_2 \Rightarrow [L(e_3)]_B=\begin{bmatrix}0\\1\end{bmatrix}$

$\Rightarrow A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}$

Ex: Find the representation matrix A for the differential operator $D: \underbrace{P_3 = \text{span}\{x^2, x, 1\}}_{E} \to \underbrace{P_2 = \text{span}\{x, 1\}}_{F}$.

$$A = \left[[D(x^2)]_F \; [D(x)]_F \; [D(1)]_F \right]$$

$$= \left[[2x]_F \; [1]_F \; [0]_F \right] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

check $[D(ax^2+bx+c)]_F = [2ax+b]_F = \begin{bmatrix} 2a \\ b \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Ex: $D: P_3 \to P_3$. $D(p(x)) = p'(x)$, $S = \{x^2, x, 1\}$ a basis for P_3.

1) Find the representation matrix A;

2) Find $\text{Ker}(D)$;

3) Find $\mathcal{R}(D)$.

1) $A = \left[[D(x^2)]_S \; [D(x)]_S \; [D(1)]_S \right] = \left[[2x]_S \; [1]_S \; [0]_S \right] = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

2) $\text{Ker}(D)$. $D(ax^2+bx+c) = 2ax+b = 0 \Rightarrow a = 0, b = 0$.

$\Rightarrow \text{Ker}(D) = \text{span}\{1\}$.

or find $N(A)$ first. $A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix}$ $x_3 = t$.

$N(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \Leftrightarrow \text{Ker}(D) = \text{span}\{1\}$.

3) Since $D(ax^2+bx+c) = 2ax+b \Rightarrow \mathcal{R}(D) = \text{span}\{x, 1\}$.

Ex. Consider $P_3 = \text{span}\{x^2, x, 1\}$. Let $S = \{x^2, x, 1\}$.

$B = \{x^2, x-1, x+1\}$. Define an operator

$$T(p(x)) = p''(x) + (x+2)p'(x) - p(x)$$

1) Find the representation matrix A of T w.r.t. S and B;

2) $Ker(T)$;

3) $R(T)$.

1) $A = \left[[T(x^2)]_B \; [T(x)]_B \; [T(1)]_B \right] =$

$T(x^2) = 2 + (x+2)2x - x^2 = x^2 + 4x + 2 = ax^2 + b(x-1) + c(x+1)$

$\Rightarrow a = 1,\ b = 1,\ c = 3.\ \Rightarrow [T(x^2)]_B = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

$T(x) = 0 + (x+2)1 - x = 2 = ax^2 + b(x-1) + c(x+1) \Rightarrow \begin{matrix} a = 0 \\ b = -1 \\ c = 1 \end{matrix}$

$\Rightarrow [T(x)]_B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$T(1) = 0 + (x+2)\cdot 0 - 1 = -1 = ax^2 + b(x-1) + c(x+1) \Rightarrow \begin{matrix} a = 0 \\ b = 1/2 \\ c = -1/2. \end{matrix}$

$[T(1)]_B = \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1/2 \\ 3 & 1 & -1/2 \end{bmatrix}.$

2). To find $Ker(T)$, first find $N(A)$.

$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1/2 \\ 3 & 1 & -1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1/2 \\ 0 & 1 & -1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_3 = t \\ x_2 = 1/2\, t \end{matrix}$

$N(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix} \right\} \Rightarrow Ker(T) = \text{span}\left\{ 0x^2 + \frac{1}{2}x + 1 \right\}.$

check $T(\frac{1}{2}x + 1) = 0 + (x+2)\frac{1}{2} - (\frac{1}{2}x + 1) = 0.$

3) To find $\mathcal{R}(T)$, we use $CS(A)$.

$$A=\begin{bmatrix}1&0&0\\1&-1&1/2\\3&1&-1/2\end{bmatrix}\to\cdots\to\begin{bmatrix}1&0&0\\0&1&-1/2\\0&0&0\end{bmatrix} \quad CS(A)=\text{span}\left\{\begin{bmatrix}1\\1\\3\end{bmatrix},\begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$$

$$\Rightarrow \mathcal{R}(T)=\text{span}\left\{\begin{matrix}1x^2+1(x-1)+3(x+1)=x^2+4x+2,\\ 0x^2-1(x-1)+1(x+1)=2\end{matrix}\right\}$$

$$=\text{span}\{x^2+4x+2,\ 2\}.$$

When $L:\mathbb{R}^n\to\mathbb{R}^m$, $V=\{v_1,\cdots,v_n\}$, $U=\{u_1,\cdots,u_m\}$.

$$A=\left[[L(v_1)]_U\ \cdots\ [L(v_n)]_U\right].$$

Ex: Let $L:\mathbb{R}^2\to\mathbb{R}^3$ be given by $L(x)=(x_2,x_1+x_2,x_1-x_2)^T$.

$$=\begin{bmatrix}0&1\\1&1\\1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}.$$

w.r.t. $S_2=\{e_1,e_2\}$, $S_3=\{e_1,e_2,e_3\}$.

1) Find the representation matrix A w.r.t. $U=\{u_1,u_2\}$ for $\mathbb{R}^2$, and $B=\{b_1,b_2,b_3\}$ for $\mathbb{R}^3$, where

$$u_1=\begin{bmatrix}1\\2\end{bmatrix},\ u_2=\begin{bmatrix}3\\1\end{bmatrix},\ b_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\ b_2=\begin{bmatrix}1\\1\\0\end{bmatrix},\ b_3=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

2) Find $\text{Ker}(L)$.

3) Find $\mathcal{R}(L)$.

1) $$A=\left[[L(u_1)]_B\ [L(u_2)]_B\right]=\begin{bmatrix}-1&-3\\4&2\\-1&2\end{bmatrix}.$$

$$L(u_1)=\begin{bmatrix}2\\1+2\\1-2\end{bmatrix}=\begin{bmatrix}2\\3\\-1\end{bmatrix}=a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}1\\1\\0\end{bmatrix}+c\begin{bmatrix}1\\1\\1\end{bmatrix}\Rightarrow \begin{matrix}c=-1,\ b=4\\ a=-1.\end{matrix}\Rightarrow [L(u_1)]_B=\begin{bmatrix}-1\\4\\-1\end{bmatrix}$$

$$L(u_2)=\begin{bmatrix}1\\3+1\\3-1\end{bmatrix}=\begin{bmatrix}1\\4\\2\end{bmatrix}=a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}1\\1\\0\end{bmatrix}+c\begin{bmatrix}1\\1\\1\end{bmatrix}\Rightarrow \begin{matrix}c=2\\ b=2\\ a=-3\end{matrix}\Rightarrow [L(u_2)]_B=\begin{bmatrix}-3\\2\\2\end{bmatrix}$$

2) To find $Ker(L)$, we find $N(A)$. $A = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix} \to \begin{bmatrix} -1 & -3 \\ 0 & -10 \\ 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow x_2 = 0 \Rightarrow x_1 = 0 \Rightarrow N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$. $\Rightarrow Ker(L) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

3). To find $R(L)$, we find $CS(A) = span\left\{ \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix} \right\}$.

$$\Rightarrow R(L) = span\{-1b_1 + 4b_2 - 1b_3,\ -3b_1 + 2b_2 + 2b_3\}$$

$$= span\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \right\}.$$

That is, $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}_B = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}_B = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$.