Subject §5.2

In $\mathbb{R}^n$, $x=(x_1;\cdots;x_n)^T$, $y=(y_1;\cdots;y_n)^T$. $x \perp y \Leftrightarrow x^Ty = x_1y_1+\cdots+x_ny_n = 0$.

Given $z_1, z_2, \cdots, z_m$ in $\mathbb{R}^n$. Then

$x \perp z_1,\cdots,z_m \Rightarrow x \perp c_1z_1+\cdots+c_mz_m \Leftrightarrow x \perp \operatorname{span}\{z_1,\cdots,z_m\}$.

Def: Let $X$ and $Y$ be subsets of $\mathbb{R}^n$.

$X$ and $Y$ are said to be orthogonal if

$$x^Ty = 0 \ (x \perp y) \quad \forall\ x \in X \text{ and } y \in Y.$$

Ex: $\{e_1,\cdots,e_n\} \subset \mathbb{R}^n$. $e_i^Te_j = 0 \quad \forall\ i \neq j$.

If $X$ and $Y$ are spanned by different $e_i$'s

i.e., $X \cap Y = \{0\}$, then $X \perp Y$.

* If $X \perp Y$, then $X \cap Y = \{0\}$.

Let $z \in X \cap Y \Rightarrow z \in X$ and $z \in Y$.

$X \perp Y \Rightarrow z \perp z \Rightarrow z = 0$.

Def: Let $S$ be a set of $\mathbb{R}^n$. The set of all vectors in $\mathbb{R}^n$ that are orthogonal to $S$ is

$$S^\perp = \{x \in \mathbb{R}^n : x^Ty = 0,\ \forall\ y \in S\},$$

called the orthogonal complement of $S$.

THM. $S^\perp$ is a subspace of $\mathbb{R}^n$.

Proof: Let $x$ and $y$ be in $S^\perp$, $\alpha, \beta \in \mathbb{R}$.

For any $z \in S$, we have $x^T z = 0$, $y^T z = 0$

then $(\alpha x + \beta y)^T z = \alpha x^T z + \beta y^T z = \alpha 0 + \beta 0 = 0$.

$\Rightarrow \alpha x + \beta y \in S^\perp$.

* How to find $S^\perp$ if $S = \{u_1, \cdots, u_m\}$ in $\mathbb{R}^n$, $m < n$?

Put $A_{n \times m} = [u_1 \cdots u_m]$, solve $A^T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \theta$ for $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$.

$S^\perp = N(A^T)$.

Ex: $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$. To find $S^\perp$, put $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$

Solve $A^T x = \theta \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}$ $\begin{matrix} x_3 = t \\ x_2 = \frac{1}{2} t \\ x_1 = -2t. \end{matrix}$

$S^\perp = N(A^T) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$ *.

consider $\mathbb{R}^3$, $\{e_1, e_2, e_3\}$. If $X = \operatorname{span}\{e_1\}$, $X^\perp = \operatorname{span}\{e_2, e_3\}$.

If $X = \operatorname{span}\{e_1, e_3\}$, $X^\perp = \operatorname{span}\{e_2\}$

consider $Lx = Ax$, $A_{m \times n}$ $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

$R(L) = R(A) = \{ b \in \mathbb{R}^m : b = Ax \text{ for some } x \in \mathbb{R}^n \} = CS(A)$.

$R(A^T) = \{ y \in \mathbb{R}^n : y = A^T x \text{ for some } x \in \mathbb{R}^m \} = RS(A)$.

$N(A) = \{ x \in \mathbb{R}^n : Ax = \theta \}$. $N(A^T) = \{ y \in \mathbb{R}^m : A^T y = \theta \}$.

THM (Fundamental Subspace Theorem) For $A_{m\times n}$,

$$N(A) = R(A^T)^{\perp}, \quad N(A^T) = R(A)^{\perp}.$$

Proof: For any $z \in R(A^T)$, $z = A^T y$ for some $y \in \mathbb{R}^m$:

$\forall x \in \mathbb{R}^n$, $x^T z = x^T A^T y = (Ax)^T y$.

$\forall x \in N(A) \Leftrightarrow Ax = 0 \Leftrightarrow x^T z = 0, \forall z \in R(A^T) \Leftrightarrow z \in N(A)^{\perp}$. #

THM: If $S$ is a subspace of $\mathbb{R}^n$, then $\dim S + \dim S^{\perp} = n$.

Furthermore if $\{a_1, \cdots, a_r\}$ is a basis for $S$ and $\{a_{r+1}, \cdots, a_n\}$ is a basis for $S^{\perp}$, then $\{a_1, \cdots, a_r, a_{r+1}, \cdots, a_n\}$ is a basis for $\mathbb{R}^n$.

Proof: If $S = \{0\}$, then $S^{\perp} = \mathbb{R}^n$. $0 + n = n$.

If $S \neq \{0\}$ and $\{a_1, \cdots, a_r\}$ is a basis for $S$. Put

$$A^T = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ a_{21} & & a_{2r} \\ \vdots & & \vdots \\ a_{n1} & & a_{nr} \end{bmatrix} \quad \begin{aligned} S &= CS(A^T) = R(A^T) \\ S^{\perp} &= R(A^T)^{\perp} = N(A). \end{aligned}$$

Since $r = \operatorname{rank}(A) = \operatorname{rank}(A^T) = \dim(R(A^T)) = \dim(S)$.

$k = \dim(N(A)) = \dim(S^{\perp})$.

$n = r + k \Rightarrow \dim(S) + \dim(S^{\perp}) = n$.

To show $\{a_1, \cdots, a_r, a_{r+1}, \cdots, a_n\}$ is a basis for $\mathbb{R}^n$, we only have to show it is LI. (Since $n = n$)

set $\underbrace{c_1 a_1 + \cdots + c_r a_r}_{y} + \underbrace{c_{r+1} a_{r+1} + \cdots + c_n a_n}_{z} = 0$.

Denote $y + z = 0$

$y \in S$, $z \in S^{\perp}$, $y + z = \theta \Rightarrow y = -z \in S^{\perp}$

$\Rightarrow y \in S \cap S^{\perp} = \{\theta\} \Rightarrow y = z = \theta$

$\Rightarrow c_1 a_1 + \cdots + c_r a_r = \theta$, $\quad c_{r+1} a_{r+1} + \cdots + c_n a_n = \theta$

But $\{a_1, \cdots, a_r\}$ $\{a_{r+1}, \cdots, a_n\}$ LI $\Rightarrow c_1 = \cdots = c_r = 0$
$c_{r+1} = \cdots = c_n = 0$

$\Rightarrow \{a_1, \cdots, a_n\}$ LI. #

Def. If $U$ and $V$ are subspaces of a vector space $W$, and each $w \in W$ can be uniquely expressed as
$w = u + v$ $\quad u \in U$ and $v \in V$,

then we say that $W$ is a direct sum of $U$ and $V$.

$$W = U \oplus V.$$

THM: (orthogonal decomposition) If $S$ is a subspace of $\mathbb{R}^n$, then $\mathbb{R}^n = S \oplus S^{\perp}$.

Proof: If $S = \{\theta\}$, then $S^{\perp} = \mathbb{R}^n$.

If $S \neq \{\theta\}$, $\{a_1, \cdots, a_r\}$ is a basis for $S$ and $\{a_{r+1}, \cdots, a_n\}$ is a basis for $S^{\perp}$, then $\{a_1, \cdots, a_r, a_{r+1}, \cdots, a_n\}$ is a basis for $\mathbb{R}^n$.

For any $x \in \mathbb{R}^n$, $x$ can be uniquely written as

$$x = c_1 a_1 + \cdots + c_r a_r + c_{r+1} a_{r+1} + \cdots + c_n a_n = u + v.$$

To show $u$ and $v$ are unique, if $x = u_1 + v_1$, then
$u + v = u_1 + v_1 \Rightarrow u - u_1 = v_1 - v \in S \cap S^{\perp} = \{\theta\} \Rightarrow$ $u - u_1 = \theta$, $u = u_1$, $v_1 - v = \theta$, $v = v_1$.

#.

<u>THM</u>: If $S$ is a subspace of $\mathbb{R}^n$, then $(S^{\perp})^{\perp} = S$. #.

$$\Rightarrow. \quad N(A)^{\perp} = \left(R(A^T)^{\perp}\right)^{\perp} = R(A^T).$$

$$N(A^T)^{\perp} = \left(R(A)^{\perp}\right)^{\perp} = R(A).$$

$$\Rightarrow \text{For } A_{m\times n}, \quad \underbrace{\begin{aligned} \mathbb{R}^n &= N(A) \oplus R(A^T), \\ \mathbb{R}^m &= N(A^T) \oplus R(A). \end{aligned}} \qquad \#$$

<u>Ex</u>: For $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$

Find the bases for $N(A)$, $R(A^T)$, $N(A^T)$ and $R(A)$, resp.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\left.\begin{aligned} &R(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\right\}. \quad R(A^T) = RS(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}. \\ &\text{For } N(A),\ x_3 = t,\ x_2 = -t,\ x_1 = -t,\ N(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\} \end{aligned}\right\} \perp$$

For $N(A^T)$, $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. $\begin{aligned} x_3 &= t. \\ x_2 &= -2t \\ x_1 &= -t \end{aligned}$. $N(A^T) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right\}$

Note $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \perp \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. $\qquad \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \not\perp \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. $\qquad$ ※.

For $A_{m\times n}$, $\mathbb{R}^n = N(A) \oplus R(A^T)$. $\forall x \in \mathbb{R}^n$

$x = y + z$ with $y \in N(A)$, $z \in R(A^T)$.

$\Rightarrow Ax = Ay + Az = 0 + Az = Az$.

$\Rightarrow R(A) = \{Ax : x \in \mathbb{R}^n\} = \{Az : z \in R(A^T)\}$.

If we restrict the domain of $A$ to $R(A^T)$, then

$A$ maps $R(A^T)$ onto $R(A)$.

Thus we have $A: N(A) \to \theta$, $A: R(A^T) \to R(A)$ $\underset{\text{1-to-1}}{\overset{\text{onto}}{.}}$

To see 1-to-1, let $x_1, x_2 \in R(A^T)$. $Ax_1 = Ax_2 \Rightarrow A(x_1 - x_2) = \theta$

$$\Rightarrow x_1 - x_2 \in N(A) \cap \underset{N(A)^\perp}{\underset{\shortparallel}{R(A^T)}} = \{\theta\} \Rightarrow x_1 = x_2 \Rightarrow \text{1-to-1}.$$

✱: Every matrix $A$ is invertible from $R(A^T)$ to $R(A)$.

<u>Ex.</u> $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$. $A^T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$. $R(A^T) = \text{span}\{e_1, e_2\}$.

$\Rightarrow N(A) = R(A^T)^\perp = \text{span}\{e_3\}$.

$\forall x \in \mathbb{R}^3$. $x = y + z$, with $y \in N(A)$, $z \in R(A^T)$.

$x = (x_1, x_2, x_3)^T \Rightarrow y = (0, 0, x_3)^T$, $z = (x_1, x_2, 0)^T$.

If we restrict the domain of $A$ to $R(A^T) = \text{span}\{e_1, e_2\}$

then $\forall z \in R(A^T)$, $z = (z_1, z_2, 0)^T \Rightarrow Az = (2z_1, 3z_2)$

$\Rightarrow R(A) = \mathbb{R}^2$. $A: R(A^T) \to R(A)$ is $\underset{\text{onto}}{\overset{\text{1-to-1}}{}} \Rightarrow$ invertible

$A^{-1}: R(A) \to R(A^T)$ is defined by $\begin{bmatrix} 2z_1 \\ 3z_2 \end{bmatrix} \to \begin{bmatrix} z_1 \\ z_2 \\ 0 \end{bmatrix}$. or

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} \frac{1}{2}x_1 \\ \frac{1}{3}x_2 \\ 0 \end{bmatrix}$. thus $A^{-1}_{R(A)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1/3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2\, x_1 \\ 1/3\, x_2 \\ 0 \end{bmatrix}$ ✱.