Subject §5.4.

For $x, y, z \in \mathbb{R}^n$,

1) $x^Tx \geq 0$. $x^Tx = 0 \Leftrightarrow x = \theta$.
2) $x^Ty = y^Tx$.
3) $(\alpha x + \beta y)^T z = \alpha x^T z + \beta y^T z$.

Generalization.

Def: An inner product on a vector space V is a function on $V \times V$ that assigns to each pair of vectors x and y in V a real number $\langle x, y \rangle$ such that

1) $\langle x, x \rangle \geq 0$ for all $x \in V$, and $\langle x, x \rangle = 0 \Leftrightarrow x = \theta$,
2) $\langle x, y \rangle = \langle y, x \rangle$,
3) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$.

Ex: For $\mathbb{R}^n$. $\langle x, y \rangle = x^Ty$ is an inner product.
If $\omega_i \geq 0$, $i = 1, 2, \cdots, n$, then $\langle x, y \rangle = \sum_{i=1}^{n} \omega_i x_i y_i$ is an inner product, $(w_1, \cdots, w_n)$ is called weights.

Ex: For $\mathbb{R}^{m\times n}$. $A = (a_{ij})$. $B = (b_{ij}) \in \mathbb{R}^{m\times n}$, then $\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$ is an inner product.

Ex. For $C[a,b]$, given $w(x) > 0$ continuous weight function.

$\forall f, g \in C[a,b]$, $\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\,dx$

is an inner product.

1) $\langle f, f \rangle = \int_a^b w(x) f^2(x)\,dx \geq 0 \quad \forall f \in C[a,b]$.

$\langle f, f \rangle = \int_a^b w(x) f^2(x)\,dx = 0 \Rightarrow f(x) \equiv 0$. Since $f \in C[a,b]$.

if $f(x_0) \neq 0$ for some $x_0 \in (a,b)$, then there is an interval $(x_0 - \delta, x_0 + \delta) \subset (a,b)$ such that $f(x) \neq 0$ when $x \in (x_0 - \delta, x_0 + \delta)$, then $\int_a^b w(x) f^2(x)\,dx \geq \int_{x_0-\delta}^{x_0+\delta} w(x) f^2(x)\,dx > 0$.

2) $\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\,dx = \langle g, f \rangle$.

3) $\langle \alpha f + \beta g, h \rangle = \int_a^b w(x) (\alpha f(x) + \beta g(x)) h(x)\,dx$

$$= \alpha \int_a^b w(x) f(x) h(x)\,dx + \beta \int_a^b w(x) g(x) h(x)\,dx$$

$$= \alpha \langle f, h \rangle + \beta \langle g, h \rangle.$$

Def. A vector space V with an inner product $\langle \cdot, \cdot \rangle$ is called an inner product space.

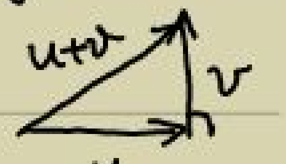
If V is an IPS, $u, v \in V$ are orthogonal if

$$\langle u, v \rangle = 0 \quad (u \perp v).$$

The length of $v \in V$ is $\|v\| = \sqrt{\langle v, v \rangle}$.

THM. (The Pythagorean Law). In a IPS V, if $u \perp v$

then $$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$



Proof. $\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \overset{=0}{\langle u,v\rangle} + \overset{=0}{\langle v,u\rangle} + \langle v,v\rangle$

$= \|u\|^2 + \|v\|^2$ ※.

Ex. $1 \perp x$ in $C[-1,1]$.

$\langle 1, x\rangle = \int_{-1}^{1} 1\cdot x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0. \Rightarrow 1 \perp x.$

$\|1+x\|^2 = \|x\|^2 + \|1\|^2$, $\quad \|x\|^2 = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}.$

$= \frac{2}{3} + 2. \quad \|1\|^2 = \int_{-1}^{1} 1^2 dx = 2.$

odd function: $f(-x) = -f(x)$. $\Rightarrow \int_{-a}^{a} f(x)dx = 0.$

even function: $f(-x) = f(x)$. $\Rightarrow \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$

Ex. Consider $C[-\pi,\pi]$. Define $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$,
i.e., $w(x) = \frac{1}{\pi}$.

Note $\sin x$ is odd and $\cos x$ is even.

$\langle \sin x, \cos x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\sin x \cdot \cos x}_{\text{odd}} dx = 0$

$\langle \sin x, \sin x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 x\,dx = \frac{2}{\pi}\int_{0}^{\pi} \frac{1-\cos 2x}{2}dx$

$= \frac{2}{\pi}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\Big|_{0}^{\pi} = 1.$

$\langle \cos x, \cos x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x\,dx = \frac{2}{\pi}\int_{0}^{\pi} \frac{1+\cos 2x}{2}dx$

$= \frac{2}{\pi}\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right)\Big|_{0}^{\pi} = 1.$

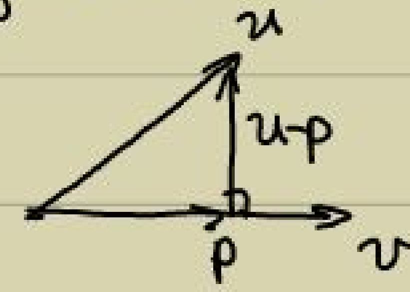
thus $\sin x, \cos x$ are orthogonal unit vectors in $C[-\pi,\pi]$.

$\|\sin x + \cos x\| = \sqrt{\|\sin x\|^2 + \|\cos x\|^2} = \sqrt{1+1} = \sqrt{2}.$ ※

Def. Let V be an IPS, $u, v \in V$ and $v \neq 0$. Then

$$\alpha = \frac{\langle u, v\rangle}{\|v\|} = \text{scalar projection of } u \text{ onto } v.$$

$$p = \frac{\langle u, v\rangle}{\|v\|}\frac{v}{\|v\|} = \text{vector projection of } u \text{ onto } v.$$



verify $u - p \perp p$.

$$\langle u-p, p\rangle = \langle u - \frac{\langle u,v\rangle}{\|v\|}\frac{v}{\|v\|}, \frac{\langle u,v\rangle v}{\|v\|^2}\rangle$$

$$= \frac{\langle u,v\rangle\langle u,v\rangle}{\|v\|^2} - \frac{\langle u,v\rangle\langle u,v\rangle\overbrace{\langle v,v\rangle}^{=\|v\|^2}}{\|v\|^4} = 0.$$

* (Cauchy–Schwarz Inequality). Let V be an IPS. u, v in V. Then

$$|\langle u, v\rangle| \leq \|u\|\|v\|,$$

where "$=$" holds if and only if u and v are LD.

So
$$-1 \leq \frac{\langle u, v\rangle}{\|u\|\|v\|} \leq 1$$

there is a unique angle θ in $[0, \pi]$ such that

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|}.$$

θ is called the angle between u and v.

In an IPS V, denote $\|v\| = \sqrt{\langle v, v\rangle}$. Verify

1) $\|v\| \geq 0 \ \forall v \in V$ and $\|v\| = 0 \Leftrightarrow v = \theta$;

2) $\|\alpha v\| = |\alpha| \|v\|$.

3) $\|u+v\| \leq \|u\| + \|v\|$ (triangle inequality).

we have $\|u+v\|^2 = \langle u+v, u+v\rangle = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2$

by CS-Inequality $\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$ ✱.

Def: A vector space V is called a normed space if there is a function, called norm, $\|\cdot\|: V \to \mathbb{R}_+$ s.t.

1) $\|v\| \geq 0 \ \forall v \in V$ and $\|v\| = 0 \Leftrightarrow v = \theta$;

2) $\|\alpha v\| = |\alpha| \|v\|$;

3) $\|u+v\| \leq \|u\| + \|v\|$.

Ex An IPS is a NS with $\|v\| = \sqrt{\langle v, v\rangle}$.

we have $\underset{\text{scalar}}{\overset{\text{dot}}{}}$ product $\Rightarrow$ Inner product $\Rightarrow$ normed space.

Ex: Consider $\mathbb{R}^n$. $x = (x_1, \cdots, x_n)$. Define different norms.

$\|x\|_2 = (\sum_{i=1}^n x_i^2)^{1/2} = \sqrt{\langle x, x\rangle}$ Inner product. 2-norm

$\|x\|_1 = \sum_{i=1}^n |x_i|$ 1-norm. $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. p-norm $1 \leq p < \infty$

$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$. $\overset{\text{max}}{\infty}$-norm. not inner product if $p \neq 2$.

Ex: In $\mathbb{R}^3$, $x=(4,-5,3)^T$.

$\|x\|_\infty = 5$, $\|x\|_1 = 4+5+3=12$, $\|x\|_2 = (16+25+9)^{1/2} = 5\sqrt{2}$.

Def: Let V be a NS. $u, v \in V$. The distance from u to v is the number $\|u-v\|$.

Ex: In $\mathbb{R}^3$, $x=(4,-5,3)^T$, $y=(1,2,3)^T$.

$\|x-y\|_2 = \sqrt{3^2+7^2+0^2} = \sqrt{58}$.

$\|x-y\|_1 = 3+7+0 = 10$. $\|x-y\|_\infty = \max\{3,7,0\} = 7$.

Ex: In $C[a,b]$. $w(x)>0$ weight.

$\|f\|_1 = \int_a^b |f(x)|dx$. $\int_a^b w(x)|f(x)|dx$

$\|f\|_2 = \left(\int_a^b f^2(x)dx\right)^{1/2}$ ~ Inner product. $\left(\int_a^b w(x)f^2(x)dx\right)^{1/2}$.

$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}$ $1 \le p < \infty$ $\left(\int_a^b w(x)|f(x)|^p dx\right)^{1/p}$.

$\|f\|_\infty = \max_{a\le x\le b} |f(x)|$.