Subject §5.5

Def: A subset S in an IPS X is orthogonal if every two distinct vectors in S are orthogonal.

THM: orthogonal nonzero vectors $\{v_1, \cdots, v_n\}$ are LI.

Proof. set $c_1 v_1 + \cdots + c_n v_n = 0$

take $\langle v_i, \cdot \rangle$ both sides. $\langle v_i, v_j \rangle = 0 \quad i \neq j$

$\Rightarrow c_i \underbrace{\langle v_i, v_i \rangle}_{\neq 0} = 0 \Rightarrow c_i = 0 \quad i = 1, \cdots, n.$

Def: orthonormal = orthogonal + (norm = 1).

$$\langle u_i, u_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Ex: In $\mathbb{R}^n$, $\quad \langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$

THM: If $\{u_1, \cdots, u_n\}$ is an orthonormal basis for an IPS V.

then

$$v = \sum_{i=1}^{n} \langle u_i, v \rangle u_i \quad \forall v \in V.$$

i.e; $[v]_U = \begin{bmatrix} \langle u_1, v \rangle \\ \langle u_2, v \rangle \\ \vdots \\ \langle u_n, v \rangle \end{bmatrix}$ or $v = c_1 u_1 + \cdots + c_n u_n$ with $c_i = \langle u_i, v \rangle, i = 1, \cdots, n$

$$\langle u, v \rangle = \sum_{i=1}^{n} \langle u_i, u \rangle \langle u_i, v \rangle. \quad \forall u, v \in V.$$

Def: An $n \times n$ matrix Q is said to be orthonormal if the column vectors of Q are orthonormal set in $\mathbb{R}^n$.

THM. Q is orthonormal $\Leftrightarrow Q^TQ = I \Leftrightarrow Q^{-1} = Q^T$.

Proof: $Q = [q_1, \cdots, q_n]$. $Q^TQ = (q_i^T q_j)$ where $q_i^T q_j = \delta_{ij}$.

Ex: The rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthonormal

$$R_\theta^T R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = I.$$

$$R_\theta^{-1} = R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R_\theta^T.$$

If Q is $\perp$-normal matrix, then

$\langle Qx, Qy \rangle = (Qy)^T Qx = y^T Q^T Q x = y^T x = \langle x, y \rangle$.

i.e., Inner products are preserved under Q-multiplication.

$\Rightarrow \|Qx\|_2 = \|x\|_2 \Rightarrow$ norm is also preserved.

THM: If Q is $\perp$-normal, then

1) Column vectors of Q form an $\perp$-normal basis for $\mathbb{R}^n$;

2) $Q^TQ = I$;

3) $Q^{-1} = Q^T$;

4) $\langle Qx, Qy \rangle = \langle x, y \rangle$

5) $\|Qx\|_2 = \|x\|_2$.

If $S = \text{span}\{u_1, \cdots, u_n\}$, then $v \perp S \Leftrightarrow v \perp u_i, i = 1, \cdots, n$.

Least square problem: $\underset{m \times n}{A} x = b$

To find a least square solution, solve the normal equation

$$A^TAx = A^Tb. \quad (A^{-1} \text{ is not defined})$$

$p = A\bar{x}$ = projection of b onto $\mathcal{R}(A)$.

If the column vectors of A are $\perp$-normal, we have

$$A^TA = I.$$

So the normal equation becomes

$$\hat{x} = A^Tb.$$

The proj of b onto $\mathcal{R}(A)$ is $p = A\hat{x} = AA^Tb$.

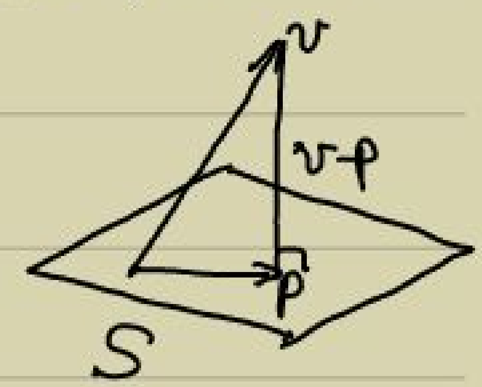
THM. Let $\{u_1, \cdots, u_n\}$ be an $\perp$-normal basis for a subspace S of an IPS V and $v \in V$.

Then the projection of v onto S is

$$p = \sum_{i=1}^{n} \langle u_i, v \rangle u_i .$$

A necessary & sufficient condition that p is the proj of v onto S is $v - p \in S^{\perp}$

or $v - p \perp u_i, \ i = 1, 2, \cdots, n.$



Proof: 1) Proj $\Rightarrow \perp$.

$$\langle u_j, p - v \rangle = \langle u_j, p \rangle - \langle u_j, v \rangle = \langle u_j, \sum_{i=1}^{n} \langle u_i, v \rangle u_i \rangle - \langle u_j, v \rangle$$

$$= \underbrace{\langle u_j, u_j \rangle}_{=1} \langle u_j, v \rangle - \langle u_j, v \rangle = 0.$$

2). $\perp \Rightarrow$ proj.

$$\| v - u \|^2 \overset{?}{>} \| v - p \|^2 \quad \forall u \in S, \ u \neq p$$

$$= \| v - p + p - u \|^2 \xlongequal{v-p \perp p-u} \| v - p \|^2 + \| p - u \|^2 > \| v - p \|^2. \checkmark$$

Write $p = c_1 u_1 + \cdots + c_n u_n$. $v - p \perp u_j, \ j = 1, 2, \cdots, n$

$$0 = \langle u_j, v - \sum_{i=1}^{n} c_i u_i \rangle = \langle u_j, v \rangle - c_j \underbrace{\langle u_j, u_j \rangle}_{=1} \Rightarrow c_j = \langle u_j, v \rangle.$$

$$\Rightarrow p = \sum_{i=1}^{n} \langle u_i, v \rangle u_i . \qquad \#.$$

Corollary: Let $\{u_1, \cdots, u_k\}$ be an $\perp$-normal basis for a subspace S of $\mathbb{R}^n$ and $v \in \mathbb{R}^n$. Then the proj of v onto S is

$$p = \sum_{i=1}^{k} \langle u_i, v \rangle u_i = U U^T v \quad \text{where } U = [u_1 \cdots u_k]_{n \times k}$$

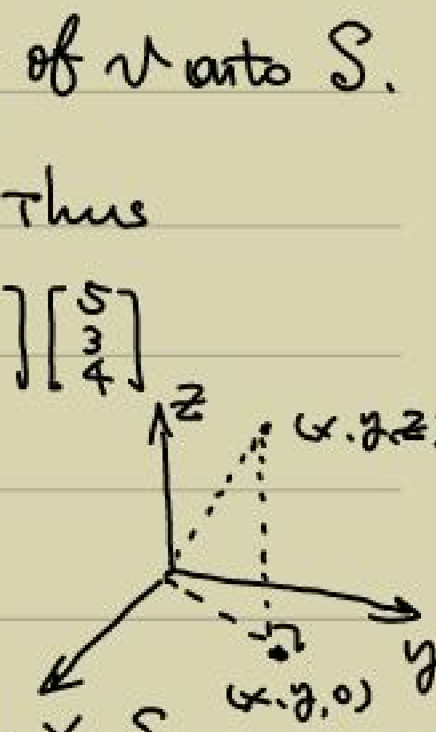
Proof: $U U^T v = [u_1, \cdots, u_k][u_1^T v \cdots u_k^T v] = \sum_{i=1}^{k} (u_i^T v) u_i = \sum_{i=1}^{k} \langle u_i, v \rangle u_i . \quad \#.$

Ex: In $\mathbb{R}^3$, $S=\left\{\begin{bmatrix}x\\y\\0\end{bmatrix}\right\}$, $v=\begin{bmatrix}5\\3\\4\end{bmatrix}$. Find the proj of v onto S.

An $\perp$-normal basis for S is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$, Thus

$$p=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}^T\begin{bmatrix}5\\3\\4\end{bmatrix}=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}5\\3\\4\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}5\\3\\4\end{bmatrix}=\begin{bmatrix}5\\3\\0\end{bmatrix}.$$



$* \; p=\langle e_1,v\rangle e_1+\langle e_2,v\rangle e_2=5\begin{bmatrix}1\\0\\0\end{bmatrix}+3\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}5\\3\\0\end{bmatrix}.$

Approximation of functions.

Consider $C[a,b]$, $\langle f,g\rangle=\int_a^b f(x)g(x)\,dx$, $\|f\|=\sqrt{\langle f,f\rangle}$.

Let $\{f_1,\cdots,f_n\}$ be an $\perp$-normal basis for a subspace S of $C[a,b]$, Given $g\in C[a,b]$. Find proj p of g onto S.

$$\|p-g\|<\|f-g\| \quad \forall f\in S,\ f\neq p.$$

where
$$p=\sum_{i=1}^{n}\langle f_i,g\rangle f_i$$

$$p-g\perp S \text{ or } p-g\perp f_i,\ i=1,2,\cdots,n.$$

polynomials

Approximation by ~~trigonometric~~ polynomials of degree n.

$$\begin{cases} c_0+c_1x+\cdots+c_nx^n \\ \dfrac{a_0}{2}+\displaystyle\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)\end{cases}$$

Recall

THM: Let $\{u_1, u_2, \cdots, u_n\}$ be a basis for a subspace S of an IPS V and $v \in V$. The proj p of v onto S is given by

$$v-p \perp S \iff v-p \perp u_i, \; i=1,\cdots,n. \quad *$$

Write $p = \sum_{j=1}^{n} c_j u_j$. $v-p \perp S \iff v-p \perp u_i$

$$\iff \langle v, u_i \rangle = \langle p, u_i \rangle = \sum_{j=1}^{n} c_j \langle u_j, u_i \rangle, \quad i=1,2,\cdots,n$$

Let
$$A = \begin{bmatrix} \langle u_1,u_1\rangle & \langle u_2,u_1\rangle & \cdots & \langle u_n,u_1\rangle \\ \langle u_1,u_2\rangle & \langle u_2,u_2\rangle & \cdots & \langle u_n,u_2\rangle \\ \cdots & & & \\ \langle u_1,u_n\rangle & \langle u_2,u_n\rangle & \cdots & \langle u_n,u_n\rangle \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_n \end{bmatrix}, \quad D = \begin{bmatrix} \langle v,u_1\rangle \\ \langle v,u_2\rangle \\ \vdots \\ \langle v,u_n\rangle \end{bmatrix}.$$

Then $v-p \perp S \iff AC = D$ ··· normal equation.

If $\{u_1, \cdots, u_n\}$ is an $\perp$-normal basis, then $A = I$, $C = D$ and $p = \sum_{j=1}^{n} \langle v, u_j \rangle u_j$ = proj of v onto S.

Ex: Find the best least square approximation of e^x on $[0,1]$ by a linear function, $p = c_0 + c_1 x$.

1) $S = \text{span}\{1, x\}$. $\{1, x\}$ is a basis for S but not $\perp$-normal.

2) check $\{1, \sqrt{12}(x-\frac{1}{2})\}$ is an $\perp$-normal basis for S.

1) normal equation: $\begin{bmatrix} \langle 1,1\rangle & \langle 1,x\rangle \\ \langle x,1\rangle & \langle x,x\rangle \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \langle e^x,1\rangle \\ \langle e^x,x\rangle \end{bmatrix}$

$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$. $\Rightarrow \langle 1,1\rangle = 1$, $\langle 1,x\rangle = \frac{1}{2}$, $\langle x,x\rangle = 1/3$.

$\langle e^x, 1\rangle = e-1$, $\langle e^x, x\rangle = 1$.

the normal equation becomes $\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} e-1 \\ 1 \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \frac{1}{\frac{1}{3}-\frac{1}{4}}\begin{bmatrix} 1/3 & -1/2 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} e-1 \\ 1 \end{bmatrix} = 12\begin{bmatrix} \frac{1}{3}(e-1)-1/2 \\ 1-\frac{1}{2}(e-1) \end{bmatrix} = \begin{bmatrix} 4e-10 \\ 18-6e \end{bmatrix}$$

$p = 4e-10+(18-6e)x$.

2) $p = \langle 1, e^x\rangle 1 + \langle \sqrt{12}(x-\frac{1}{2}), e^x\rangle \sqrt{12}(x-\frac{1}{2})$

$\langle 1, e^x\rangle = e-1$

$\langle x, e^x\rangle = 1, \Rightarrow \langle \sqrt{12}(x-\frac{1}{2}), e^x\rangle = \sqrt{12}\langle x, e^x\rangle - \frac{\sqrt{12}}{2}\langle 1, e^x\rangle$

$= \sqrt{12} - \sqrt{3}(e-1)$

$$\Rightarrow p = e-1+(\sqrt{12}-\sqrt{3}(e-1))\sqrt{12}(x-\tfrac{1}{2})$$
$$= e-1-(\sqrt{12}-\sqrt{3}(e-1))\sqrt{3}+(\sqrt{12}-\sqrt{3}(e-1))\sqrt{12}\,x$$
$$= 4e-10+(18-6e)x.$$

#

$\{\frac{1}{\sqrt{2}}, \sin x, \cos x, \sin 2x, \cos 2x, \cdots, \sin nx, \cos nx\}$ forms an $\perp$-normal set in $C[-\pi,\pi]$ with the inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx.$$

the best approximation of $f \in C[-\pi,\pi]$ by trig-polynomial of degree n or less can be obtained by

$p(f) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{n}(a_k\cos kx + b_k \sin kx)$ -- Fourier Approximation

with $a_0 = \langle f, \frac{1}{\sqrt{2}}\rangle \frac{1}{\sqrt{2}} = \langle f, 1\rangle \frac{1}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$

$a_k = \langle f, \cos kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx$

$b_k = \langle f, \sin kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx$

a_0, a_k, b_k, Fourier Coefficients.