§5.6. Gram–Schmidt Orthogonalization Process

Subject

Given a basis $\{x_1, \cdots, x_n\}$, try to find an orthonormal basis $\{u_1, \cdots, u_n\}$ from $\{x_1, \cdots, x_n\}$ such that

$$\text{span}\{x_1, \cdots, x_k\} = \text{span}\{u_1, \cdots, u_k\}, \quad k = 1, 2, \cdots, n.$$

Step 1: Let $u_1 = x_1 / \|x_1\|$. $\Rightarrow \|u_1\| = 1$, and $\text{span}\{x_1\} = \text{span}\{u_1\}$

Step 2: Let $p_1 = \langle x_2, u_1 \rangle u_1 =$ proj of x_2 onto $\text{span}\{u_1\} \Rightarrow x_2 - p_1 \perp u_1$.

set $u_2 = \dfrac{x_2 - p_1}{\|x_2 - p_1\|}$ $\Rightarrow u_2 \perp u_1$, $\|u_2\| = 1$, $\text{span}\{x_1, x_2\} = \text{span}\{u_1, u_2\}$

Step 3: Let $p_2 = \langle x_3, u_1 \rangle u_1 + \langle x_3, u_2 \rangle u_2 =$ proj of x_3 onto $\text{span}\{u_1, u_2\}$.

$\Rightarrow x_3 - p_2 \perp u_1, u_2$, set $u_3 = \dfrac{x_3 - p_2}{\|x_3 - p_2\|}$.

$\Rightarrow \|u_3\| = 1$. and $\text{span}\{x_1, x_2, x_3\} = \text{span}\{u_1, u_2, u_3\}$

$\cdots$

Step k : Assume we have found orthonormal set $\{u_1, \cdots, u_k\}$ such that $\text{span}\{x_1, \cdots, x_r\} = \text{span}\{u_1, \cdots, u_r\}$, $r = 1, 2, \cdots, k$.

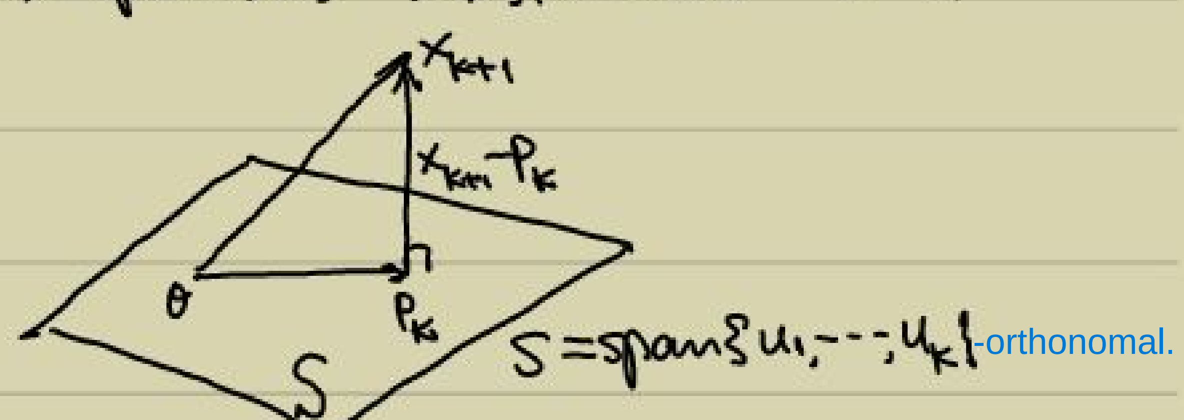
Step (k+1): Compute $p_k = \langle x_{k+1}, u_1 \rangle u_1 + \cdots + \langle x_{k+1}, u_k \rangle u_k$

$= $ proj of x_{k+1} onto $\text{span}\{u_1, \cdots, u_k\}$

$\Rightarrow x_{k+1} - p_k \perp u_1, \cdots, u_k$. set $u_{k+1} = \dfrac{x_{k+1} - p_k}{\|x_{k+1} - p_k\|}$

we have $\|u_{k+1}\| = 1$, $u_{k+1} \perp u_1, \cdots, u_k$,

$\text{span}\{x_1, \cdots, x_r\} = \text{span}\{u_1, \cdots, u_r\}$, $r = 1, 2, \cdots, k+1$.



Ex 1. Find an orthonormal basis for $P_3[-1,1] = \text{span}\{1, x, x^2\}$.

$\{1, x, x^2\}$ is a basis.

Step 1. $\|1\|^2 = \int_{-1}^{1} 1^2 dx = 2$. $\|1\| = \sqrt{2}$. $u_1 = \frac{1}{\|1\|} = \frac{1}{\sqrt{2}}$.

Step 2. $p_1 = \langle x, u_1 \rangle u_1 = 0$, $\langle x, \frac{1}{\sqrt{2}} \rangle = \int_{-1}^{1} x \frac{1}{\sqrt{2}} dx = 0$ odd.

$\|x\|^2 = \int_{-1}^{1} x^2 dx = \frac{2}{3}$. $u_2 = \frac{x}{\|x\|} = \sqrt{\frac{3}{2}} x$.

Step 3. $p_2 = \langle x^2, u_1 \rangle u_1 + \langle x^2, u_2 \rangle u_2$.

$\langle x^2, u_1 \rangle = \int_{-1}^{1} x^2 \frac{1}{\sqrt{2}} dx = \frac{2}{3} \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{3}$.

$\langle x^2, u_2 \rangle = \int_{-1}^{1} x^2 \frac{\sqrt{3}}{\sqrt{2}} x \, dx = 0$. odd

$p_2 = \langle x^2, u_1 \rangle u_1 = \frac{\sqrt{2}}{3} \frac{1}{\sqrt{2}} = \frac{1}{3}$.

$\|x^2 - p_2\|^2 = \int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx = \int_{-1}^{1} (x^4 - \frac{2}{3}x^2 + \frac{1}{9}) dx = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}$

$u_3 = \frac{x^2 - p_2}{\|x^2 - p_2\|} = \sqrt{\frac{45}{8}} (x^2 - \frac{1}{3})$.

the orthonormal basis is $\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} x, \frac{3}{4}\sqrt{10} (x^2 - \frac{1}{3}) \right\}$.

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Ex: Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$. Find an $\perp$-normal basis for $R(A)$.

$\quad\quad\quad a_1 \quad a_2 \quad a_3$

Sol: Set $r_{11} = \|a_1\| = 2$, $q_1 = \frac{1}{r_{11}} a_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$;

$r_{12} = \langle a_2, q_1 \rangle = q_1^T a_2 = -\frac{1}{2} + 2 + 2 - \frac{1}{2} = 3$.

$p_1 = \langle a_2, q_1 \rangle q_1 = r_{12} q_1 = 3 q_1$;

$$a_2 - p_1 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix};$$

$$r_{22} = \|a_2 - p_1\| = \sqrt{4(5/2)^2} = 5.$$

$$q_2 = \frac{1}{\|a_2 - p_1\|}(a_2 - p_1) = \frac{1}{r_{22}}(a_2 - p_1) = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix};$$

$$r_{13} = \langle a_3, q_1 \rangle = 2 - 1 + 1 = 2,$$

$$r_{23} = \langle a_3, q_2 \rangle = -2 - 1 + 1 = -2$$

$$p_2 = \langle a_3, q_1 \rangle q_1 + \langle a_3, q_2 \rangle q_2 = r_{13} q_1 + r_{23} q_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix};$$

$$a_3 - p_2 = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}, \qquad r_{33} = \|a_3 - p_2\| = \sqrt{4 * 2^2} = 4.$$

$$q_3 = \frac{1}{\|a_3 - p_2\|}(a_3 - p_2) = \frac{1}{4}\begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}:$$

Set

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$ an upper triangular matrix.

$$Q = [q_1\ q_2\ q_3] = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}$$

Then

$$A = QR \quad \text{or} \quad \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

THM. (QR Factorization)

If $A = [a_1 \cdots a_n]$ is an $m \times n$ matrix of rank n, then

$$A_{m\times n} = Q_{m\times n} R_{n\times n}$$

where Q has orthonormal columns and

R is upper triangular

with $r_{11} = \|a_1\|$, $r_{kk} = \|a_k - p_{k-1}\|$, $r_{ik} = q_i^T a_k$, $k = i+1, \cdots, n$;

p_k = projection of a_{k+1} onto span$\{q_1, \cdots, q_k\}$.

$q_1 = \dfrac{a_1}{\|a_1\|}$, $q_{k+1} = \dfrac{a_{k+1} - p_k}{\|a_{k+1} - p_k\|}$.

Least square solution to $Ax = b$

normal equation $A^T A x = A^T b$

So $A = QR$, then $(QR)^T (QR) x = (QR)^T b = R^T Q^T b$

$$R^T Q^T Q R x = R^T R x$$

$\Rightarrow R^T R x = R^T Q^T b$, $(R^T)^{-1} \Rightarrow R x = Q^T b \Rightarrow x = R^{-1} Q^T b$.

THM. If $A_{m\times n}$ has rank n, then the least square solution to $Ax = b$ is given by

1) $\hat{x} = R^{-1} Q^T b$ from backward substitution $Rx = Q^T b$.

where $A = QR$ is the QR factorization.