Subject §6.1 Eigenvalues

Let $A_{n\times n}$ be a sequare matrix. If there is a scalar λ and a nonzero vector $x \in \mathbb{R}^n$ such that

$$Ax = \lambda x$$

then λ is called an eigenvalue of A and x is called an eigenvector of A corresponding to the eigenvalue λ.

For one eigenvalue λ, the set of all eigenvectors of A w.r.t. λ is called the eigenspace of A w.r.t. λ.

Remarks:

1) E-vector $x \neq \theta$, $\quad A\theta = \lambda\theta \quad \forall \lambda$.

2) If $Ax = \lambda x$, then $A(\alpha x) = \lambda(\alpha x) \quad \forall \alpha \in \mathbb{R}$.

   If $Ax_1 = \lambda x_1$, $Ax_2 = \lambda x_2$, then $A(\alpha x_1 + \beta x_2) = \lambda(\alpha x_1 + \beta x_2)$.

   $\Rightarrow$ eigenspace is a vector space.

How to find E-values and corresponding E-vectors?

Note: $Ax = \lambda x \Leftrightarrow (A - \lambda I)x = \theta \quad (x \neq \theta) \Rightarrow A - \lambda I$ is singular.

$\Leftrightarrow \det(A - \lambda I) = 0$, called the characteristic equation, from which we can solve for E-values $\lambda_1, \cdots, \lambda_k$.

then for each λ_i, $i=1,2,\cdots,K$, we have

$$\det(A-\lambda_i I)=0 \iff N(A-\lambda_i I) \neq \{0\}.$$

For each $x_i \in N(A-\lambda_i I)$ and $x_i \neq 0$, it holds

$$Ax_i = \lambda_i x_i.$$

$x_i \sim$ E-vector w.r.t. λ_i.

* $p(\lambda) = \det(A-\lambda I)$ is called the characteristic polynomial.

<u>Ex.</u> $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$. Find all E-values and E-vectors

1) set characteristic equation:

$$|A-\lambda I| = \det\begin{pmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{pmatrix} = (\lambda-3)(\lambda+2)-6 = 0$$

2) Find all the roots: $|A-\lambda I| = \lambda^2-\lambda-6-6 = \lambda^2-\lambda-12 = 0$

a) complete square: $(\lambda-4)(\lambda+3)=0 \Rightarrow \lambda_1=-3,\ \lambda_2=4$;

or b) quadratic formula: $\lambda_{1,2} = \dfrac{1 \pm \sqrt{1^2-4\cdot(-12)}}{2} = \dfrac{1}{2} \pm \dfrac{7}{2} = \begin{cases} -3 \\ 4 \end{cases}$.

3) a) For $\lambda_1=-3$, solve $(A-\lambda I)x=0$ for x.

$\Rightarrow A-\lambda I = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ always proportional since $|A-\lambda I|=0$ $\Rightarrow x = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

actually let $x_1=t$, $x_2=-3t$. $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. drop t.

b) For $\lambda_2=4$, solve $(A-\lambda_2 I)X=0$ for X.

$A-\lambda_2 I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \to \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$ two rows are proportional $\Rightarrow X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} -b \\ a \end{bmatrix}$ or $\begin{bmatrix} b \\ -a \end{bmatrix}$.

Ex: Find all E-values and E-vectors of

$$A=\begin{bmatrix}2&-3&1\\1&-2&1\\1&-3&2\end{bmatrix}.$$

1) set char. equation:

$$0=\det(A-\lambda I)=\det\left(\begin{bmatrix}2&-3&1\\1&-2&1\\1&-3&2\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)=\begin{vmatrix}2-\lambda&-3&1\\1&-2-\lambda&1\\1&-3&2-\lambda\end{vmatrix}$$

$$=\left|\begin{matrix}2-\lambda&-3&1\\1&-2-\lambda&1\\1&-3&2-\lambda\end{matrix}\right|\begin{matrix}2-\lambda&-3\\1&-2-\lambda\\1&-3\end{matrix}\Bigg|=\begin{matrix}-(2+\lambda)(2-\lambda)^2-3-3\\+(2+\lambda)+3(2-\lambda)+3(2-\lambda)\end{matrix}$$

2) Find roots:

$$=-(2+\lambda)(\lambda^2-2\lambda+4)-6-5\lambda+14=-(\lambda^3-2\lambda^2-4\lambda+8)+8-5\lambda$$

$$=-\lambda^3+2\lambda^2-\lambda=-\lambda(\lambda^2-2\lambda+1)=-\lambda(\lambda-1)^2=0$$

$\lambda_1=0$, $\lambda_2=1$ double

3) For $\lambda_1=0$, $A-\lambda I=A=\begin{bmatrix}2&-3&1\\1&-2&1\\1&-3&2\end{bmatrix}\to\begin{bmatrix}1&-2&1\\2&-3&1\\1&-3&2\end{bmatrix}\to\begin{bmatrix}1&-2&1\\0&1&-1\\0&-1&1\end{bmatrix}\to\begin{bmatrix}1&-2&1\\0&1&-1\\0&0&0\end{bmatrix}$

always at least a row of zeros since $|A-\lambda I|=0$.

$x_2=t \Rightarrow x_3=t$, $x_1=t$. $X_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$.

For $\lambda_2=1$, $A-\lambda_2 I=\begin{bmatrix}1&-3&1\\1&-3&1\\1&-3&1\end{bmatrix}\to\begin{bmatrix}1&-3&1\\0&0&0\\0&0&0\end{bmatrix}$ set $x_2=t$, $x_3=s$ $\Rightarrow x_1=3t-s$

$X=t\begin{bmatrix}3\\1\\0\end{bmatrix}+s\begin{bmatrix}-1\\0\\1\end{bmatrix}$, $X_2=\begin{bmatrix}3\\1\\0\end{bmatrix}$. $X_3=\begin{bmatrix}-1\\0\\1\end{bmatrix}$ two linearly indep E-vectors

Eigenspace of A for $\lambda_2=1=\mathrm{span}\left\{\begin{bmatrix}3\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$.

Complex E-values, complex numbers. $i=\sqrt{-1}$, $i^2=-1$. $c=a+ib$, $\bar{c}=a-ib$

$a\sim$ real part, $b\sim$ imaginary part. $\bar{c}\sim$ complex conjugate of c.

Ex. Given $A=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}$. Find all E-values and E-vectors.

$$|A-\lambda I|=\begin{vmatrix}1-\lambda & 2\\ -2 & 1-\lambda\end{vmatrix}=(1-\lambda)^2+4=\lambda^2-2\lambda+5=0,\quad \lambda_{1,2}=\frac{2\pm\sqrt{4-4\cdot 5}}{2}=1\pm i2$$

For $\lambda_1=1+i2$.

$$A-\lambda_1 I=\begin{bmatrix}-i2 & 2\\ -2 & -i2\end{bmatrix}\xrightarrow{(2)=-i(1)}\begin{bmatrix}-i2 & 2\\ 0 & 0\end{bmatrix}\to\begin{bmatrix}-i & 1\\ 0 & 0\end{bmatrix}\quad \begin{matrix}x_1=t\\ x_2=it\end{matrix}\quad X_1=\begin{bmatrix}1\\0\end{bmatrix}+i\begin{bmatrix}0\\1\end{bmatrix}.$$

For $\lambda_2=1-i2$.

$$A-\lambda_2 I=\begin{bmatrix}i2 & 2\\ -2 & i2\end{bmatrix}\xrightarrow{(2)=i(1)}\begin{bmatrix}i2 & 2\\ 0 & 0\end{bmatrix}\to\begin{bmatrix}i & 1\\ 0 & 0\end{bmatrix}\quad \begin{matrix}x_1=t\\ x_2=-it\end{matrix}\quad X_2=\begin{bmatrix}1\\0\end{bmatrix}-i\begin{bmatrix}0\\1\end{bmatrix}.$$

$\lambda_2=\overline{\lambda_1}$, $X_2=\overline{X_1}$. complex conjugate pairing.

THM: If $A_{n\times n}$ is a real matrix, then its complex E-values and E-vectors must appear in conjugate pairs, i.e.,
if $\lambda_1=a+ib$ then $\lambda_2=a-ib$
similarly if $Z=X+iY$ is an E-vector w.r.t. $\lambda=a+ib$
then $\overline{Z}=X-iY$ is an E-vector w.r.t. $\overline{\lambda}=a-ib$.

Proof: If $AZ=\lambda Z$, then $\overline{AZ}=\overline{\lambda Z}\Leftrightarrow A\overline{Z}=\overline{\lambda}\,\overline{Z}$.

THM: (The product and sum of E-values) Let $A_{n\times n}=(a_{ij})$. and $\lambda_1,\cdots,\lambda_n$ be E-values of A. Denote $p(\lambda)=\det(A-\lambda I)$

Then 1) $\prod_{i=1}^{n}\lambda_i=\lambda_1\cdots\cdots\lambda_n=p(0)=\det(A)$;

2). $\sum_{i=1}^{n}\lambda_i=\lambda_1+\cdots+\lambda_n=a_{11}+a_{22}+\cdots+a_{nn}=\sum_{i=1}^{n}a_{ii}=\mathrm{tr}(A)$ (trace).

Ex: $A = \begin{bmatrix} 5 & -18 \\ 1 & -1 \end{bmatrix}$. $\det(A) = -5+18 = 13$. $\operatorname{tr}(A) = 5-1 = 4$

So $\lambda_1 \cdot \lambda_2 = 13$, $\lambda_1 + \lambda_2 = 4$

$\Downarrow$

$\lambda_2 = 13/\lambda_1$. $\quad \lambda_1 + 13/\lambda_1 = 4 \Rightarrow \lambda_1^2 - 4\lambda_1 + 13 = 0 \quad (*)$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 4\cdot 13}}{2} = 2 \pm i3.$$

check $(2+i3)\cdot(2-i3) = 2^2 + 3^2 = 13$, $\quad 2+i3 + (2-i3) = 4$.

On the other hand: $|A - \lambda I| = \begin{vmatrix} 5-\lambda & -18 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda-5)(\lambda+1)+18$

$= \lambda^2 - 4\lambda + 13 = 0 \quad (*)$

Thus $\lambda_{1,2} = 2 \pm i3$ are E-values.

Properties

1) If $Ax = \lambda x$, then $A^2 x = \lambda^2 x, \cdots, A^k x = \lambda^k x$.

2). Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$

$P(A) = a_n A^n + \cdots + a_n A + a_0 I$.

then $Ax = \lambda x \Rightarrow P(A)x = P(\lambda)x$.

3) A is singular $\Leftrightarrow \lambda = 0$ is an E-value;

4) A is nonsingular. $Ax = \lambda x \Rightarrow A^{-1}x = \lambda^{-1}x$.

(multiply both sides from right by $\lambda^{-1}A^{-1}$).

Recall: $B_{n\times n}$ and $A_{n\times n}$ are said to be similar if there is a nonsingular matrix $S_{n\times n}$ such that

$B = S^{-1}AS \quad (B \sim A \Rightarrow A \sim B.\ B \sim A, C \sim B \Rightarrow C \sim A)$.

THM: If $B_{n\times n}$ is similar to $A_{n\times n}$, then they have the same characteristic polynomial and consequently the same E-values. (not E-vectors)

Proof: Denote $P_B(\lambda) = \det(B-\lambda I)$ and $P_A(\lambda) = \det(A-\lambda I)$. the characteristic polynomials of B and A resp.
If B is similar to A, there is a nonsingular matrix S such that $B = S^{-1}AS$. Thus

$$P_B(\lambda) = \det(B-\lambda I) = \det(S^{-1}AS-\lambda I)$$
$$= \det(S^{-1}(A-\lambda I)S) = \det(S^{-1})\det(A-\lambda I)\det(S)$$
$$= \det(A-\lambda I) = P_A(\lambda),$$

since $\det(S^{-1})\cdot\det(S) = 1$

Hence $P_B(\lambda)$ and $P_A(\lambda)$ have the same roots that are E-values of B and A.

Ex: Let $T = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $S = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$.

$$A = S^{-1}TS = \frac{1}{10-9}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 6 & 6 \end{bmatrix}.$$

For T, $\lambda_1 = 2$, $\lambda_2 = 3$.

For A, $\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & -2 \\ 6 & 6-\lambda \end{vmatrix} = (\lambda-6)(\lambda+1)+12$
$$= \lambda^2 - 5\lambda + 6$$
$$= (\lambda-2)(\lambda-3) = 0.$$

$\Rightarrow \lambda_1 = 2$, $\lambda_2 = 3$. The same as T.

THM. If $A_{n\times n}$ is symmetric, then E-vectors w.r.t. different E-values are orthogonal.

Proof. Let $Ax_i = \lambda_i x_i$, $Ax_j = \lambda_j x_j$ with $\lambda_i \neq \lambda_j$

Then $x_i^T A x_j = x_i^T (\lambda_j x_j) = \lambda_j x_i^T x_j$

$= (Ax_i)^T x_j = \lambda_i x_i^T x_j$.

$\lambda_i \neq \lambda_j \Rightarrow x_i^T x_j = 0$ i.e., $x_i \perp x_j$.

Ex. $A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$. $|A - \lambda I| = \begin{vmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 2$

$= \lambda^2 - 5\lambda + 4 = 0$

$= (\lambda-1)(\lambda-4)$.

$\lambda_1 = 1$, $\lambda_2 = 4$.

For $\lambda_1 = 1$. $A - \lambda_1 I = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \to \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$ $X_1 = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

$\lambda_2 = 4$, $A - \lambda_2 I = \begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ \sqrt{2} & -1 \end{bmatrix}$. $X_2 = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$.

$X_1^T X_2 = -\sqrt{2} + \sqrt{2} = 0 \Rightarrow X_1 \perp X_2$.