Subject § 6.2 Systems of linear differential equations

consider a system of first order linear differential equations with constant coefficients of the form

$$(*) \begin{cases} \frac{dy_1}{dt} = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ \frac{dy_2}{dt} = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ \cdots \\ \frac{dy_n}{dt} = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{cases}$$

where each y_i is an unknown function of t to be determined and a_{ij}'s are given constants.

Denote

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \frac{dY}{dt} = Y' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}.$$

Then $(*) \iff Y' = AY$.

It is clear that if Y_1 and Y_2 are two solutions, then so is $\alpha Y_1 + \beta Y_2$ $\quad \forall \alpha, \beta$.

If $Y_1, \cdots, Y_n$ are n LI solutions, then the general solution to $(*)$ is

$$Y = C_1 Y_1 + C_2 Y_2 + \cdots + C_n Y_n$$

where $C_1, C_2, \cdots, C_n$ are arbitrary constants, determined by n initial conditions

$y_1(0) = y_1^0,\ y_2(0) = y_2^0, \cdots, y_n(0) = y_n^0$ or $Y(0) = Y_0 = (y_1^0, y_2^0, \cdots, y_n^0)^T$.

Since for the scalar differential equation

$$y' = ay.$$

we have $y = y_0 e^{at}$.

$\left(\frac{y'}{y} = a, \ \int \frac{y'}{y} dt = \int a\, dt \Rightarrow \ln|y(t)| = at + C_0 \right.$

$\Rightarrow y(t) = e^{at + C_0} = Ce^{at}.\ \ y(0) = y_0 \Rightarrow C = y_0.$

$\left. \Rightarrow y(t) = y_0 e^{at} \right).$

For $Y' = AY$, we look for solution of the form

$$Y(t) = e^{\lambda t} V$$

for some scalar λ and nonzero vector $V \in \mathbb{R}^n$.

To find λ and V, we plug $Y(t) = e^{\lambda t} V$ into the system $Y' = AY$, we obtain $Y' = \lambda e^{\lambda t} V = A e^{\lambda t} V$

i.e., $$AV = \lambda V.$$

Thus λ is an eigenvalue of A and $V \neq 0$ is an eigenvector of A w.r.t. λ.

If $\lambda_1, \cdots, \lambda_n$ are E-values and E-vectors $V_1, \cdots, V_n$ are LI, then the general solution is

$$Y = C_1 Y_1 + C_2 Y_2 + \cdots + C_n Y_n$$

where $Y_i = e^{\lambda_i t} V_i$, $i = 1, 2, \cdots, n$, and $C_1, C_2, \cdots, C_n$ can be solved by the initial condition $Y(0) = Y_0$. (Initial-value problem)

Ex. Solve the system $\begin{cases} y_1' = 3y_1 + 4y_2, & y_1(0) = -1, \\ y_2' = 3y_1 + 2y_2 & y_2(0) = 2. \end{cases}$

we have $Y' = AY$ where $Y = [y_1, y_2]^T$. $A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$

$Y(0) = Y_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

$Y(t) = e^{\lambda t} V$. $\Leftrightarrow$ $AV = \lambda V$.

$$0 = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 12 = \lambda^2 - 5\lambda + 6 - 12$$
$$= \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1)$$

$\lambda_1 = -1$, $\lambda_2 = 6$.

For $\lambda_1 = -1$, $A - \lambda_1 I = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $Y_1(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For $\lambda_2 = 6$, $A - \lambda_2 I = \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \Rightarrow V_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. $Y_2(t) = e^{6t} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

The general solution is

$$Y(t) = C_1 Y_1 + C_2 Y_2 = C_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{6t} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} C_1 e^{-t} + 4C_2 e^{6t} \\ -C_1 e^{-t} + 3C_2 e^{6t} \end{bmatrix}.$$
$$= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

To determine C_1 and C_2 by inital condition, we have

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = Y(0) = \begin{bmatrix} C_1 + 4C_2 \\ -C_1 + 3C_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{3+4} \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -11/7 \\ 1/7 \end{bmatrix}$$

$$\Rightarrow Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -11/7\, e^{-t} + 4/7\, e^{6t} \\ 11/7\, e^{-t} + 3/7\, e^{6t} \end{bmatrix}.$$

complex eigenvalues:

Say $\lambda = a + ib$ then $\bar{\lambda} = a - ib$

$\downarrow$

$v = V_R + iV_I$, $\quad \bar{v} = V_R - iV_I$

The corresponding solutions

$Y = e^{(a+ib)t}(V_R + iV_I)$ are complex-valued

$\bar{Y} = e^{(a-ib)t}(V_R - iV_I)$ thus not interested.

To generate real-valued solutions from Y and $\bar{Y}$,

write $\quad e^{ibt} = \cos bt + i\sin bt$,

$\quad e^{-ibt} = \cos bt - i\sin bt$.

Then $Y = e^{at}(\cos bt + i\sin bt)(V_R + iV_I)$

$= e^{at}(\cos bt\, V_R - \sin bt\, V_I) + i(\cos bt\, V_I + \sin bt\, V_R)$

$\bar{Y} = e^{at}(\cos bt - i\sin bt)(V_R - iV_I)$

$= e^{at}(\cos bt\, V_R - \sin bt\, V_I) - i(\cos bt\, V_I + \sin bt\, V_R)$

Thus

$Y_1 = \frac{1}{2}Y + \frac{1}{2}\bar{Y} = e^{at}(\cos bt\, V_R - \sin bt\, V_I)$

$Y_2 = \frac{1}{2i}Y - \frac{1}{2i}\bar{Y} = e^{at}(\cos bt\, V_I + \sin bt\, V_R)$

are two real-valued solutions and will be used.

Ex: Solve the differential system $\begin{cases} y_1' = y_1 + y_2 \\ y_2' = -2y_1 + 3y_2 \end{cases}$.

$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$.

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2 = \lambda^2 - 4\lambda + 5 = 0$

$\lambda_{1,2} = \dfrac{4 \pm \sqrt{4^2 - 4\times 5}}{2} = 2 \pm i1.$

For $\lambda_1 = 2 + i1$. $A - \lambda_1 I = \begin{bmatrix} -1-i & 1 \\ -2 & 1-i \end{bmatrix}$ proportional $(-1-i)\times(1-i) = 2)$.

$x_1 = t$. $x_2 = (1+i)t$, $X = t\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \Rightarrow V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$V_R + iV_I$.

$a = 2$, $b = 1$.

$Y_1 = e^{2t}\left(\cos t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = e^{2t}\begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix}$

$Y_2 = e^{2t}\left(\cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = e^{2t}\begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}$

The general solution is

$$Y = C_1 Y_1 + C_2 Y_2 = e^{2t}\begin{bmatrix} C_1 \cos t + C_2 \sin t \\ C_1(\cos t - \sin t) + C_2(\cos t + \sin t) \end{bmatrix}.$$

If initial condition $y_1(0) = -1$, $y_2(0) = 2$ or $Y(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is prescribed, then

$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = Y(0) = \begin{bmatrix} C_1 \\ C_1 + C_2 \end{bmatrix} \Rightarrow C_1 = -1$, $C_2 = 3$. The solution is

$$Y(t) = e^{2t}\begin{bmatrix} -\cos t + 3\sin t \\ -(\cos t - \sin t) + 3(\cos t + \sin t) \end{bmatrix} = \begin{bmatrix} e^{2t}(-\cos t + 3\sin t) \\ e^{2t}(2\cos t + 4\sin t) \end{bmatrix}$$

$$= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

Convert a high-order system to a first-order system.

Given a 2nd-order system

$$Y'' = A_1 Y + A_2 Y'$$

Denote $Y_1 = Y$, $Y_2 = Y' = Y_1'$. Then

$$Y'' = Y_2' = A_1 Y_1 + A_2 Y_2$$

we have $Y_1' = Y_2$
$Y_2' = A_1 Y_1 + A_2 Y_2$ $\Rightarrow \begin{bmatrix} Y_1' \\ Y_2' \end{bmatrix} = \begin{bmatrix} 0 & I \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$

a 1st-order differential system.

Ex: Solve the second-order initial-value problem

$$y_1'' = 2y_1 + y_2 + y_1' + y_2'$$
$$y_2'' = -5y_1 + 2y_2 + 5y_1' - y_2'$$

I.C. $y_1(0) = y_2(0) = y_1'(0) = 4$
$y_2'(0) = -4$.

Denote $y_3 = y_1'$, $y_4 = y_2'$. Thus $y_1'' = y_3'$, $y_2'' = y_4'$.

we have the 1st-order system

$$\begin{cases} y_1' = y_3 \\ y_2' = y_4 \\ y_3' = 2y_1 + y_2 + y_3 + y_4 \\ y_4' = -5y_1 + 2y_2 + 5y_3 - y_4 \end{cases}$$

I.C. $y_1(0) = y_2(0) = y_3(0) = 4$
$y_4(0) = -4$.

Denote $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{bmatrix} = \left[\begin{array}{cc:cc} 0 & & I & \\ \hdashline 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{array}\right]$ $\Rightarrow$ $Y' = AY$, $Y(0) = \begin{bmatrix} 4 \\ 4 \\ 4 \\ -4 \end{bmatrix} = Y_0$.

$$A-\lambda I=\begin{bmatrix}-\lambda & 0 & 1 & 0\\ 0 & -\lambda & 0 & 1\\ 2 & 1 & 1-\lambda & 1\\ -5 & 2 & 5 & -1-\lambda\end{bmatrix}\Rightarrow \lambda_1=1,\ \lambda_2=-1,\ \lambda_3=3,\ \lambda_4=-3.$$

For $\lambda_1=1$,

$$A-\lambda_1 I=\begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\\ 2 & 1 & 0 & 1\\ -5 & 2 & 5 & -2\end{bmatrix}\to\begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\\ 0 & 1 & 2 & 1\\ 0 & 2 & 0 & -2\end{bmatrix}\to\begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\\ 0 & 0 & 2 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$x_4=t,\ x_3=-t,\ x_2=t,\ x_1=-t.\quad V_1=\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}.$

For $\lambda_2=-1$,

$$A-\lambda_2 I=\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 2 & 1 & 2 & 1\\ -5 & 2 & 5 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 2 & 10 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 0 & 1 & 5 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

$x_3=t,\ x_2=-5t,$
$x_4=5t,\ x_1=-t,\quad V_2=\begin{bmatrix}-1\\-5\\1\\5\end{bmatrix}$

For $\lambda_3=3$,

$$A-\lambda_3 I=\begin{bmatrix}-3 & 0 & 1 & 0\\ 0 & -3 & 0 & 1\\ 2 & 1 & -2 & 1\\ -5 & 2 & 5 & -4\end{bmatrix}\to\begin{bmatrix}-3 & 0 & 1 & 0\\ 0 & -3 & 0 & 1\\ 0 & 1 & -4/3 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\to\begin{bmatrix}-3 & 0 & 1 & 0\\ 0 & 0 & -4 & 4\\ 0 & 1 & -4/3 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$x_1=t$
$x_3=3t$
$x_4=3t,\quad V_3=\begin{bmatrix}1\\1\\3\\3\end{bmatrix}.$
$x_2=t$

For $\lambda_4=-3$,

$$A-\lambda_4 I=\begin{bmatrix}3 & 0 & 1 & 0\\ 0 & 3 & 0 & 1\\ 2 & 1 & 4 & 1\\ -5 & 2 & 5 & 2\end{bmatrix}\to\begin{bmatrix}3 & 0 & 1 & 0\\ 0 & 3 & 0 & 1\\ 0 & 1 & 10/3 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\to\begin{bmatrix}3 & 0 & 1 & 0\\ 0 & 0 & -10 & -2\\ 0 & 1 & 10/3 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$x_1=t$
$x_3=-3t$
$x_4=15t;\quad V_4=\begin{bmatrix}1\\-5\\-3\\15\end{bmatrix}.$
$x_2=-5t$

Then the general solution is

$$Y(t)=\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=C_1e^{t}\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}+C_2e^{-t}\begin{bmatrix}-1\\5\\1\\5\end{bmatrix}+C_3e^{3t}\begin{bmatrix}1\\1\\3\\3\end{bmatrix}+C_4e^{-3t}\begin{bmatrix}1\\-5\\-3\\15\end{bmatrix}. \qquad (*)$$

I.C.

$$Y(0)=\begin{bmatrix}4\\4\\4\\-4\end{bmatrix}=C_1\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}+C_2\begin{bmatrix}-1\\5\\1\\5\end{bmatrix}+C_3\begin{bmatrix}1\\1\\3\\3\end{bmatrix}+C_4\begin{bmatrix}1\\-5\\-3\\15\end{bmatrix}=\begin{bmatrix}-1 & -1 & 1 & 1\\ 1 & 5 & 1 & -5\\ -1 & 1 & 3 & -3\\ 1 & 5 & 3 & 15\end{bmatrix}\begin{bmatrix}C_1\\C_2\\C_3\\C_4\end{bmatrix}$$

$$\left[\begin{array}{cccc|c} -1 & -1 & 1 & 1 & 4 \\ 1 & 5 & 1 & -5 & 4 \\ -1 & 1 & 3 & -3 & 4 \\ 1 & -5 & -3 & 15 & -4 \end{array}\right] \to \left[\begin{array}{cccc|c} -1 & -1 & 1 & 1 & 4 \\ 0 & 4 & 2 & -4 & 8 \\ 0 & 2 & 2 & -4 & 0 \\ 0 & -6 & -2 & 16 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -4 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 2 & 1 & -2 & 4 \\ 0 & -3 & -1 & 8 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 4 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & -1 & 2 & 4 \\ 0 & 0 & 2 & 2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 4 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & -1 & 2 & 4 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right] \quad \begin{array}{l} C_4 = 1 \\ C_3 = -2 \\ C_2 = 4. \\ C_1 = -1. \end{array}$$

We obtain

$$Y(t) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = -e^{t}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + 4e^{-t}\begin{bmatrix} -1 \\ -5 \\ 1 \\ 5 \end{bmatrix} - 2e^{3t}\begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1 \\ -5 \\ -3 \\ 15 \end{bmatrix}$$

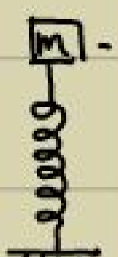
In particular

$$y_1(t) = e^{t} - 4e^{-t} - 2e^{3t} + e^{-3t},$$
$$y_2(t) = -e^{t} - 20e^{-t} - 2e^{3t} - 5e^{-3t}.$$

Actually from ⊛, the general solution to y_1 and y_2 is

$$y_1(t) = -C_1e^{t} - C_2e^{-t} + C_3e^{3t} + C_4e^{-3t}$$
$$y_2(t) = C_1e^{t} - 5C_2e^{-t} + C_3e^{3t} - 5C_4e^{-3t}.$$

Application: Spring-mass system



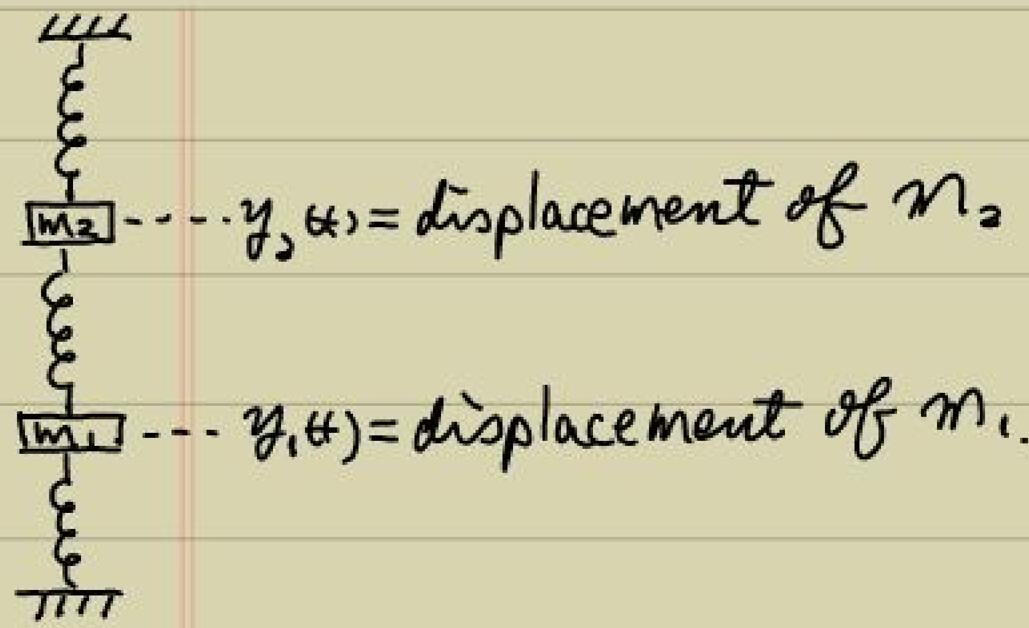
$y(t)$ = displacement from the equilibrium position at t.

m = mass

K = spring constant

We have

$$my''(t) = -Ky(t)$$



K = spring constant

we have

$$m_2 y_2''(t) = -K y_2(t) + K(y_1(t) - y_2(t)) = K y_1(t) - 2K y_2(t)$$

$$m_1 y_1''(t) = -K y_1(t) + K(y_2(t) - y_1(t)) = -2K y_1(t) + K y_2(t).$$

a 2nd-order system: $y_1''(t) = -\frac{2K}{m_1} y_1 + \frac{K}{m_1} y_2$

$$y_2''(t) = \frac{K}{m_2} y_1 - \frac{2K}{m_2} y_2.$$

Denote $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. $A = \begin{bmatrix} -\frac{2K}{m_1} & \frac{K}{m_1} \\ \frac{K}{m_2} & \frac{-2K}{m_2} \end{bmatrix}$.

Then we have $\quad Y'' = AY.$

No Y' is involved. In general, if $Y^{(m)} = AY$.

We need to compute the mth roots of the E-values of A. i.e. if λ is an E-value of A. X is an E-vector of A w.r.t. λ; σ is an mth root of λ. Then for $Y = e^{\sigma t} X$, we have

$$Y^{(m)} = \sigma^m e^{\sigma t} X = \lambda Y$$

and $\quad AY = e^{\sigma t} AX = e^{\sigma t} \lambda X = \lambda Y.$

Thus $Y = e^{\sigma t} X$ is a solution of the system.

As for the spring-mass system, if $m_1 = m_2 = k = 1$, then $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 - 1 = \lambda^2 + 4\lambda + 3 = 0$,

$\lambda_{1,2} = \dfrac{-4 \pm \sqrt{4^2 - 4\times 3}}{2} = \begin{cases} -3 \\ -1. \end{cases}$

For $\lambda_1 = -3$, $A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\lambda_2 = -1$, $A - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\sigma_1^2 = \lambda_1 = -3 \Rightarrow \sigma_1 = \pm\sqrt{3}\, i$, $\sigma_2^2 = \lambda_2 = -1, \Rightarrow \sigma_2 = \pm i$

we have 4 complex valued solutions: $e^{\sigma_1 t} X_1, e^{\sigma_2 t} X_2$.

$e^{\sqrt{3} i t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, e^{-\sqrt{3} i t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, e^{it}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{-it}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To generate real valued solutions, we take their linear combinations:

$\frac{1}{2} e^{\sqrt{3} i t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2} e^{-\sqrt{3} i t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}\left(e^{\sqrt{3} i t} + e^{-\sqrt{3} i t}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \cos\sqrt{3}t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\frac{1}{2i} e^{\sqrt{3} i t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{1}{2i} e^{-\sqrt{3} i t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2i}\left(e^{\sqrt{3} i t} - e^{-\sqrt{3} i t}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \sin\sqrt{3}t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\frac{1}{2} e^{it}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} e^{-it}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \cos t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\frac{1}{2i} e^{it}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2i} e^{-it}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sin t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

we obtain the real valued general solution

$Y(t) = (C_1 \cos\sqrt{3}t + C_2 \sin\sqrt{3}t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (C_3 \cos t + C_4 \sin t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

If I.C. $Y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $Y'(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ are given, we have

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Y(0) = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_3\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow C_1 = C_3 = 0$.

$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = Y'(0) = C_2\sqrt{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_4\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow C_2 = 0, C_4 = 2$.

$\Rightarrow Y(t) = \begin{bmatrix} 2\sin t \\ 2\sin t \end{bmatrix}$.

Frequence $= 1$. and amplitude $= 2$. *