Test 1 Examples

Tuesday, February 23, 2021 2:01 PM

§1.2 #6 (b)

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 2 & -2 & 1 & 2 & 8 \\ 3 & 1 & 2 & -1 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & -8 & -1 & 0 & 2 \\ 0 & -8 & -1 & -4 & -10 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & -8 & -1 & 0 & 2 \\ 0 & 0 & 0 & -4 & -12 \end{array}\right] \uparrow$$

$x_4 = 3$, $x_2 = s$, $x_3 = -2-8s$, $x_1 = 2+5s$

#10 $$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & a-3 & b-2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & b-4 \end{array}\right]$$

a) $a=5$, $b\neq 4$, no solution

b) $a=5$, $b=4$. infinitely many solutions

c) $a\neq 5$. Unique solution.

§1.3. #13. See homework hint.

§1.5 #10 (f)

$$\overset{A}{\left[\begin{array}{ccc|ccc} 2 & 0 & 5 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right]} \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 2 & 0 & 5 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -5 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -5 \\ 0 & 1 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right]$$

A^{-1}

§2.1 #3 (g)

$$\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix} = \left|\begin{matrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & -2 & 3 \end{matrix}\right|\begin{matrix} 2 & 0 \\ 1 & 2 \\ 1 & -2 \end{matrix}\right| = 12-2-2=8$$

or

$$= -\begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 4 & -3 \\ 0 & -2 & 4 & -5 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 5 & 4 & -3 \\ -2 & 4 & -5 \end{vmatrix} = -\begin{vmatrix} 4 & -3 \\ 4 & -5 \end{vmatrix}$$

$$= 8$$

§3.2 #12

a) Yes $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$. $3=3$.

b). Yes, first three span $\mathbb{R}^3$.

c). No. $\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{vmatrix} = 0$. $3=3$.

d) No. 2nd and 3rd are multiple of 1st.

e). $n=3$. 2 vectors not enough to span $\mathbb{R}^3$.

#16 a) No, there is not x term.

b) Yes. 1st 3 terms span $1, x, x^2$.

c) Yes. $1=(x+2)-(x+1)$, $x=2(x+1)-(x+2)$, $x^2=(x^2-1)+(x+2)-(x+1)$.

d) No. 2 vectors not enough to span P_3.

§3.3. #2

a). Yes. $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 \neq 0$.

b). No. $n=3$, $m=4>3$. by a theorem.

c) No. $n=3=m$. $\left|\begin{array}{ccc|cc} 2 & 3 & 2 & 2 & 3 \\ 1 & 2 & 2 & 1 & 2 \\ -2 & -2 & 0 & -2 & -2 \end{array}\right| = \begin{matrix} -12-4 \\ +8+8 \end{matrix} = 0$

d) No. $n=3=m$ $\begin{vmatrix} 2 & -2 & 4 \\ 1 & -1 & 2 \\ -2 & 2 & 4 \end{vmatrix} = 0$.

e) Yes. two vectors are not multiple $\Rightarrow$ LI.

#8. a) No. $x^2-2 = -2(1)+(x^2)$

b). No. $n=3<4=m$.

c) Yes. $c_1(x+2)+c_2(x+1)+c_3(x^2-1)=0$

$c_3x^2+(c_1+c_2)x+2c_1+c_2-c_3=0$

x^2. $c_3=0$

x. $c_1+c_2=0$ $\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \end{vmatrix} = 1-2 \neq 0$

1. $2c_1+c_2-c_3=0$

$\Rightarrow C_1 = C_2 = C_3 = 0 \Rightarrow LI.$

d). Yes. two vectors are not multiple $\Rightarrow$ LI.

§3.4. #8.

a). No, two vectors are not enough to span $\mathbb{R}^3$.

b). x_1, x_2, x_3 have to be LI.

c)
$$\begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & -4 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & -4 & -1 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & -5 & 1 & 4 \end{bmatrix} \Rightarrow X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\uparrow \; \uparrow \; \uparrow$$

#10
$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 5 & 3 & 7 & 1 \\ 2 & 4 & 2 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

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$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^3$.

#14. a) $x = x$, $1 = x - (x-1)$, $x^2 = (x^2+1) - x + (x-1)$

$d = 3 = \dim P_3 = \dim(\mathrm{span}\{1, x, x^2\})$

b). by a). $d=3$

c) $d=2$, since $x^2-x-1 = x^2-(x+1)$.

d). $d=2$.

§3.5 #5. $V[x]_V = U[x]_U$, $V=[e_1,e_2,e_3]$, $U=[u_1,u_2,u_3]$.

a) Find $V \xrightarrow{T} U$ i.e. $T[x]_V = [x]_U$

$\Rightarrow T = U^{-1}$

$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]}_{U} \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$U^{-1} = T$

b). i) $T\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, ii) $T\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ iii) $T=\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$.

#9. $V=\{x, 1\}$. $U=\{2x-1, 2x+1\}$.

a). $U \xrightarrow{T} V$.

$2x-1 = 2\cdot x - 1\cdot 1$

$2x+1 = 2\cdot x + 1\cdot 1$ $\Rightarrow T = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

b). $V \xrightarrow{W} U$. $\Rightarrow W = T^{-1} = \frac{1}{2+2}\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$

#10 set $E = \{1, x, x^2\}$. $U = \{1, 1+x, 1+x+x^2\}$.

Ask for $E \xrightarrow{T} U$

$$1 = 1\cdot 1 + 0\cdot(1+x) + 0(1+x+x^2)$$

$$x = -1\cdot 1 + 1\cdot(1+x) + 0(1+x+x^2)$$

$$x^2 = 0\cdot 1 - 1\cdot(1+x) + 1\cdot(1+x+x^2)$$

$$\Rightarrow \begin{array}{c c} & \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{array}{c} 1 \\ 1+x \\ 1+x+x^2 \end{array} & \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = T.$$

§3.6 #1 b)

$$\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 14 & 1 & -4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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$$RS(A) = \text{span}\left\{\begin{matrix}[-3 & 1 & 3 & 4] \\ [1 & 2 & -1 & -2] \\ [-3 & 8 & 4 & 2]\end{matrix}\right\} = \text{span}\left\{\begin{matrix}[1 & 0 & 0 & -10/7] \\ [0 & 1 & 0 & -2/7] \\ [0 & 0 & 1 & 0]\end{matrix}\right\}$$

$r = 3$

$$CS(A) = \left\{\begin{bmatrix}-3\\1\\-3\end{bmatrix}, \begin{bmatrix}1\\2\\8\end{bmatrix}, \begin{bmatrix}3\\-1\\4\end{bmatrix}\right\}$$

$4-3=1$ freedom

For $N(A)$. solve $Ax=0$. $x_4 = t$, $x_3 = 0$, $x_2 = \frac{2}{7}t$, $x_1 = \frac{10}{7}t$.

$$N(A) = \text{span}\left\{\begin{bmatrix}10/7\\2/7\\0\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}10\\2\\0\\7\end{bmatrix}\right\} \quad k=1.$$

$n = 4 = r + k = 3 + 1$.

#15. Did in Lecture Notes §3.6.

i) $x = x_0 + N(A)$

ii) $Ax_0 = b \Longleftrightarrow$

$3a_1 + 2a_2 + 0a_3 + 2a_4 + 0a_5 = b \Rightarrow a_4$

$a_5 = -a_1 - 2a_2 + 5a_4 \Rightarrow a_5$.